

SELECTIVE GAMES ON BINARY RELATIONS

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ABSTRACT. We present a unified approach, based on dominating families in binary relations, for the study of topological properties defined in terms of selection principles and the games associated to them.

INTRODUCTION

Classical games introduced by Berner and Juhász [12], Galvin [18], Gruenhage [21] and Telgársky [35, 36] associated with diverse topological properties, as well as several subsequent games, can be considered in a single unifying framework based on the notion of a *relation*, defined below in Definition 2.1, and of a *dominating family* for a relation, defined below in Definition 2.2. We define these games in Definition 3.3. This framework subsumes and clarifies several isolated theorems about topological games.¹

One of the two main phenomena about these topological games that is addressed here is: Consider a topological property, E . For many examples of E one can find spaces X and Y each of which has the property E , while the product space $X \times Y$ does not have the property E . Define a space X to be *productively* E if, for each space Y that has property E , also $X \times Y$ has the property E . For some properties E it is a significant mathematical problem to characterize the spaces X that are productively E . For several isolated examples of topological games G it has been found that, if a certain player of the game G on a space X has a winning strategy, then X is productively E .

We study the productivity of properties E in this abstract context. The four properties E we consider — which will be described in detail after Definition 3.9 — are as follows:

- E_1 : TWO has a winning strategy in the game G ;
- E_2 : ONE does not have a winning strategy in the game G ;
- E_3 : A selective version of a certain countability hypothesis holds;
- E_4 : A certain countability hypothesis holds.

For the properties we consider it will be the case that

$$E_1 \Rightarrow E_2 \Rightarrow E_3 \Rightarrow E_4,$$

where sometimes, but not always, an implication is reversible.

The nature of our theorems is as follows:

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¹Which specific theorems in the literature are affected will be presented later in the paper.

- A relation in the class \mathbf{E}_1 is productively \mathbf{E}_1 .
- A relation in the class \mathbf{E}_1 is productively \mathbf{E}_3 .
- A relation in the class \mathbf{E}_1 is productively \mathbf{E}_4 .

It is curious that as of yet the techniques for the other implications do not seem to produce the implication that a space in the class \mathbf{E}_1 is productively \mathbf{E}_2 . Some questions related to this issue will be posed at the end of the paper.

These results about products bring us to analyzing situations in which TWO has a winning strategy in a game, and the second of the two main phenomena addressed here: In many instances of games \mathbf{G} it is known that, if player TWO has a winning strategy in the game \mathbf{G} , then player TWO has a winning strategy in a game \mathbf{G}' , in which the winning condition for TWO appears more stringent.

This paper is organized as follows: After establishing notational conventions in Section 1 we introduce a general framework for the theory regarding \mathbf{E}_1 in Section 2. In Section 3 we translate classical duality results on games to the new framework. In Sections 4 and 5 we prove product theorems. Sections 6 and 7 are dedicated to situations in which the existence of a winning strategy for player TWO in a certain game turns out to be equivalent to the same condition in other games that are seemingly more difficult for TWO. In Section 8 we study conditions under which a Lindelöf-like property is equivalent to ONE not having a winning strategy in the selective game associated to the property being considered. Section 9 contains some final remarks about the results presented in the paper.

1. NOTATIONAL CONVENTIONS

Throughout our paper X denotes the underlying set of a topological space and τ denotes the ambient topology on the space. Whenever a second topological space Y is involved, its topology will be denoted by ρ . Unless explicitly stated otherwise, we do not make any assumptions about separation hypotheses on the topological spaces in our results.

For a space X and a point $x \in X$, we write $\tau_x = \{U \in \tau : x \in U\}$. For a subset A of X , the closure of A in X is denoted by \overline{A} . The set of all compact subsets of X is denoted by $K(X)$.

A set A is *countable* if $|A| \leq \aleph_0$. For a cardinal number λ , we write $[A]^{<\lambda} = \{B \subseteq A : |B| < \lambda\}$, $[A]^\lambda = \{B \subseteq A : |B| = \lambda\}$ and $[A]^{\leq \lambda} = [A]^{<\lambda} \cup [A]^\lambda$. The set of all functions from a set A to a set B is denoted by AB ; we also write ${}^{<\omega}B$ for $\bigcup_{n \in \omega} {}^nB$.

Throughout the paper we will make use of several families associated to a topological space X ; the reader is referred to Example 2.3 for both their definition and the notation we adopt to denote these families.

For definitions of concepts found in the paper that are neither listed here nor defined right before the result in which they appear, the reader is referred to [16], [23] and [26].

We end this section with two definitions we shall make use of frequently in the paper.

Definition 1.1. *Let \mathcal{A} and \mathcal{B} be families of sets. We say that $(\mathcal{A}, \mathcal{B})$ -Lindelöf holds if every element of \mathcal{A} has a countable subset that is an element of \mathcal{B} . If $\mathcal{A} = \mathcal{B}$, we will say \mathcal{A} -Lindelöf instead of $(\mathcal{A}, \mathcal{B})$ -Lindelöf.*

Definition 1.2. *Let \mathcal{A} and \mathcal{B} be families of sets.*

- The notation $S_1(\mathcal{A}, \mathcal{B})$ abbreviates the following statement:
For every sequence $(A_n)_{n \in \omega}$ of elements of \mathcal{A} , there is a sequence $(B_n)_{n \in \omega}$ such that $B_n \in A_n$ for all $n \in \omega$ and $\{B_n : n \in \omega\} \in \mathcal{B}$.
- The notation $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ abbreviates the following statement:
For every sequence $(A_n)_{n \in \omega}$ of elements of \mathcal{A} , there is a sequence $(F_n)_{n \in \omega}$ such that $F_n \in [A_n]^{<\aleph_0}$ for all $n \in \omega$ and $\bigcup_{n \in \omega} F_n \in \mathcal{B}$.

Note that $S_1(\mathcal{A}, \mathcal{B})$ implies $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$, which in turn implies $(\mathcal{A}, \mathcal{B})$ -Lindelöf.

2. RELATIONS AND DOMINATING FAMILIES

All of our main results in this paper are phrased in terms of *dominating families in binary relations*, a general framework that allows us to express a number of topological concepts through a unified terminology. This section is dedicated to stating the basic definitions and exploring how this framework can be used to capture properties of interest in general topology.

The following definition is based on [42]; see also [13, Section 4].

Definition 2.1. A relation is a triple (A, B, R) , where $A \neq \emptyset$ and $R \subseteq A \times B$ is such that $\forall a \in A \exists b \in B ((a, b) \in R)$.

We will henceforth adopt the convention of writing aRb instead of $(a, b) \in R$, and we will read this as “ b dominates a in R ”.

Definition 2.2. For a relation $P = (A, B, R)$, define

$$\text{Dom}(P) = \{Z \subseteq B : \forall a \in A \exists b \in Z (aRb)\}.$$

The elements of $\text{Dom}(P)$ are said to be dominating in P .

The table in Example 2.3 has double purpose: It serves to illustrate instances of relations whose set of dominating families have topological meaning, and also to define the concepts and terminology that are listed therein.

Example 2.3. Let X be a topological space with topology τ , and let $x \in X$. The following table lists several examples of families associated to X (and x) that can be expressed as the set $\text{Dom}(P)$ for some relation P .

P	$\text{Dom}(P)$
(X, τ, \in)	$\mathcal{O}_X = \{\mathcal{U} \subseteq \tau : X = \bigcup \mathcal{U}\}$
$([X]^{<\aleph_0}, \tau, \subseteq)$	$\Omega_X = \{\mathcal{U} \subseteq \tau : \mathcal{U} \text{ is an } \omega\text{-cover of } X\}$
$(K(X), \tau, \subseteq)$	$\mathcal{K}_X = \{\mathcal{U} \subseteq \tau : \mathcal{U} \text{ is a } k\text{-cover of } X\}$
$(K(X), K(X) \setminus \{\emptyset\}, R),$ where $HRK \leftrightarrow H \cap K = \emptyset$	$\mathfrak{M}_X = \{\mathcal{M} \subseteq K(X) \setminus \{\emptyset\} : \mathcal{M} \text{ is a moving-off family}\}$
$(\tau \setminus \{\emptyset\}, X, \ni)$	$\mathfrak{D}_X = \{D \subseteq X : X = \overline{D}\}$
$(\tau \setminus \{\emptyset\}, \tau, R),$ where $URV \leftrightarrow U \cap V \neq \emptyset$	$\mathcal{D}_X = \{\mathcal{U} \subseteq \tau : X = \overline{\bigcup \mathcal{U}}\}$
$(\tau \setminus \{\emptyset\}, \tau \setminus \{\emptyset\}, \supseteq)$	$\Pi_X = \{\mathcal{V} \subseteq \tau \setminus \{\emptyset\} : \mathcal{V} \text{ is a } \pi\text{-base for } X\}$
$(\{(a, U) : a \in U \in \tau\}, \tau, R),$ where $(a, U)RV \leftrightarrow a \in V \subseteq U$	$\mathfrak{B}_X = \{\mathcal{B} \subseteq \tau : \mathcal{B} \text{ is a base for } X\}$
$(\tau_x, \tau_x, \supseteq)$	$\mathfrak{V}_x = \{\mathcal{V} \subseteq \tau_x : \mathcal{V} \text{ is a local base for } X \text{ at } x\}$
$(X \setminus \{x\}, \tau_x, \notin)$ (assuming (X, τ) is T_1)	$\Psi_x = \{\mathcal{V} \subseteq \tau_x : \mathcal{V} \neq \emptyset \text{ and } \bigcap \mathcal{V} = \{x\}\}$
(τ_x, X, \ni)	$\Omega_x = \{A \subseteq X : x \in \overline{A}\}$
$(\tau_x, [X]^{<\aleph_0} \setminus \{\emptyset\}, \supseteq)$	$\pi\mathcal{N}_x = \{\mathcal{S} \subseteq [X]^{<\aleph_0} \setminus \{\emptyset\} : \mathcal{S} \text{ is a } \pi\text{-network at } x\}$
$(\tau_x, [X]^{\leq \aleph_0} \setminus \{\emptyset\}, \supseteq)$	$\pi\mathcal{N}_x^{\aleph_0} = \{\mathcal{S} \subseteq [X]^{\leq \aleph_0} \setminus \{\emptyset\} : \mathcal{S} \text{ is a } \pi\text{-network at } x\}$

Note that several classical topological properties can be phrased in terms of families of the form $\text{Dom}(P)$: For example, a topological space X is Lindelöf if and only if \mathcal{O}_X -Lindelöf holds, and X is countably tight at a point $x \in X$ if and only if Ω_x -Lindelöf holds.

3. GAMES AND DUALITY

We now proceed to defining the basic games we shall consider in this paper.

Definition 3.1. We say that two games G and G' are equivalent if both of the following hold:

- ONE has a winning strategy in the game G if, and only if, ONE has a winning strategy in G' ; and
- TWO has a winning strategy in the game G if, and only if, TWO has a winning strategy in G' .

Definition 3.2. We say that two games G and G' are dual games if both of the following hold:

- ONE has a winning strategy in the game G if, and only if, TWO has a winning strategy in G' ; and
- TWO has a winning strategy in the game G if, and only if, ONE has a winning strategy in G' .

The first game we consider is a natural game associated with a pair of relations:

Definition 3.3. Let $P = (A, B, R)$ and $Q = (C, D, T)$ be relations. The game $G(P, Q)$ is defined as follows: In each inning $n \in \omega$, player ONE chooses $a_n \in A$, and then player TWO chooses $b_n \in B$ with $a_n R b_n$. ONE wins a play

$$a_0, b_0, \dots, a_n, b_n, \dots$$

if $\{b_n : n \in \omega\} \in \text{Dom}(Q)$. Otherwise, TWO wins.

As the following examples show, there are several topological games studied in the literature that can be regarded as instances of the game $G(P, Q)$ introduced above.

Example 3.4. Let P be the relation (X, τ, \in) . Then $\text{Dom}(P)$ is \mathcal{O}_X , the collection of open covers of X . In this instance the game $G(P, P)$ corresponds to the point-open game $G(X)$ of Galvin, introduced in [18].

Example 3.5. Let P be the relation $(\tau \setminus \{\emptyset\}, X, \ni)$. Then $\text{Dom}(P)$ is \mathfrak{D}_X , the collection of dense subsets of X . In this instance the game $G(P, P)$ corresponds to the point-picking game $G_\omega^D(X)$ of Berner and Juhász, introduced in [12].

Example 3.6. Fix a point $x \in X$. Let P be the relation (τ_x, X, \ni) . Then $\text{Dom}(P)$ is Ω_x , the collection of subsets of X that have x as a cluster point. In this instance the game $G(P, P)$ corresponds to the game $G_{O, P}^c(X, x)$ of Gruenhage, introduced in [21].

Example 3.7. Let P be the relation (X, τ, \in) and Q be the relation $(\tau \setminus \{\emptyset\}, \tau, T)$, where $UTV \leftrightarrow U \cap V \neq \emptyset$. Then $\text{Dom}(Q)$ is \mathcal{D}_X , the collection of open families with union dense in X . In this instance the game $G(P, Q)$ corresponds to the game $\theta(X)$ of Tkachuk, introduced in [39].

Now, towards the duality theorem mentioned in the introduction, define the following game:

Definition 3.8. Let \mathcal{A} and \mathcal{B} be families of sets with $\mathcal{A} \neq \emptyset$. The game $G_1(\mathcal{A}, \mathcal{B})$ is played as follows. In each inning $n \in \omega$, ONE first chooses a set $A_n \in \mathcal{A}$, and then TWO responds with a $B_n \in A_n$. A play

$$A_0, B_0, \dots, A_n, B_n, \dots$$

is won by TWO if $\{B_n : n \in \omega\}$ is an element of \mathcal{B} . Otherwise, ONE wins.

We shall also refer to the following variation of the previous game later on.

Definition 3.9. Let \mathcal{A} and \mathcal{B} be families of sets with $\mathcal{A} \neq \emptyset$. The game $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ is played as follows. In each inning $n \in \omega$, ONE first chooses a set $A_n \in \mathcal{A}$, and then TWO responds with a finite subset F_n of A_n . A play

$$A_0, F_0, \dots, A_n, F_n, \dots$$

is won by TWO if $\bigcup_{n \in \omega} F_n$ is an element of \mathcal{B} . Otherwise, ONE wins.

It is immediate that, if TWO has a winning strategy in $G_1(\mathcal{A}, \mathcal{B})$, then TWO has a winning strategy in $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$. Similarly, if ONE has a winning strategy in $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$, then ONE has a winning strategy in $G_1(\mathcal{A}, \mathcal{B})$.

Furthermore, we have the following chain of implications (now defining in more detail the four properties E_1 – E_4 considered in the Introduction):

$$\begin{array}{c}
\text{TWO has a winning strategy in } G_1(\mathcal{A}, \mathcal{B}) \\
\Downarrow \\
\text{ONE does not have a winning strategy in } G_1(\mathcal{A}, \mathcal{B}) \\
\Downarrow \\
S_1(\mathcal{A}, \mathcal{B}) \\
\Downarrow \\
(\mathcal{A}, \mathcal{B})\text{-Lindel\"of.}
\end{array}$$

Note that the same implications hold when the subscript “1” is replaced with “fin”.

The following duality result is a rephrasing of Theorem 1 of [18].

Theorem 3.10 (Galvin [18]). *For all relations P and Q , the games $G(P, Q)$ and $G_1(\text{Dom}(P), \text{Dom}(Q))$ are dual.*

Proof. We will prove that, if TWO has a winning strategy in $G_1(\text{Dom}(P), \text{Dom}(Q))$, then ONE has a winning strategy in $G(P, Q)$. The reader shall find no difficulty in verifying the remaining three implications.

The core of the proof is the following observation: If $\sigma : (<^\omega \text{Dom}(P)) \setminus \{\emptyset\} \rightarrow B$ is a strategy for TWO in $G_1(\text{Dom}(P), \text{Dom}(Q))$, then

$$(\dagger) \quad \forall s \in <^\omega \text{Dom}(P) \exists a_s \in A (\{b \in B : a_s R b\} \subseteq \{\sigma(s^\frown(Z)) : Z \in \text{Dom}(P)\}).$$

Suppose, in order to get a contradiction, that there is $s \in <^\omega \text{Dom}(P)$ witnessing the failure of (\dagger) . Then, for each $a \in A$, there is $b_a \in B$ with $a R b_a$ such that there is no $Z \in \text{Dom}(P)$ satisfying $b_a = \sigma(s^\frown(Z))$. But then we get a contradiction from the fact that $\tilde{Z} = \{b_a : a \in A\} \in \text{Dom}(P)$, since this implies that $\sigma(s^\frown(\tilde{Z})) = b_a$ for some $a \in A$.

Having proved (\dagger) , let now $\sigma : (<^\omega \text{Dom}(P)) \setminus \{\emptyset\} \rightarrow B$ be a winning strategy for TWO in $G_1(\text{Dom}(P), \text{Dom}(Q))$. A winning strategy for ONE in $G(P, Q)$ can then be defined as follows.

ONE’s initial move is $a_\emptyset \in A$. If $b_0 \in B$ is TWO’s response, it follows from (\dagger) that there is $Z_0 \in \text{Dom}(P)$ such that $b_0 = \sigma((Z_0))$; let then $a_{(Z_0)} \in A$ be ONE’s next move in the play. TWO will respond with some $b_1 \in B$ satisfying $a_{(Z_0)} R b_1$; by (\dagger) , there is $Z_1 \in \text{Dom}(P)$ satisfying $b_1 = \sigma((Z_0, Z_1))$; ONE’s next move will then be $a_{(Z_0, Z_1)} \in A$; and so forth.

By proceeding in this fashion, we obtain a sequence $(Z_n)_{n \in \omega}$ of elements of $\text{Dom}(P)$ and a play of $G(P, Q)$ in which TWO’s move in the inning $n \in \omega$ is $b_n = \sigma((Z_k)_{k \leq n})$. Since σ is a winning strategy for TWO in $G_1(\text{Dom}(P), \text{Dom}(Q))$, it follows that $\{b_n : n \in \omega\} = \{\sigma((Z_k)_{k \leq n}) : n \in \omega\} \in \text{Dom}(Q)$; therefore, the strategy above described is a winning strategy for ONE in $G(P, Q)$. \square

Theorem 3.10 expresses in terms of relations and dominating families the underlying argument in duality results such as the following corollaries.

The first one, in which the argument above was first presented, is Theorem 1 of [18]. Recall that the *Rothberger game on X* [18] is the game $G_1(\mathcal{O}_X, \mathcal{O}_X)$. A topological space X is a *Rothberger space* [30] if $S_1(\mathcal{O}_X, \mathcal{O}_X)$ holds.

Corollary 3.11 (Galvin [18]). *For every topological space X , the point-open game on X and the Rothberger game on X are dual games.*

Proof. Apply Theorem 3.10 with $P = Q = (X, \tau, \in)$. \square

The next corollary is the combination of Theorems 7 and 8 of [33].

Corollary 3.12 (Scheepers [33]). *For every topological space X , the games $G_\omega^D(X)$ and $G_1(\mathfrak{D}_X, \mathfrak{D}_X)$ are dual games.*

Proof. Apply Theorem 3.10 with $P = Q = (\tau \setminus \{\emptyset\}, X, \ni)$. \square

The next corollary is Theorem 3.3(2) of [39].

Corollary 3.13 (Tkachuk [39]). *For a topological space X , the games $\theta(X)$ and $G_1(\mathcal{O}_X, \mathcal{D}_X)$ are dual games.*

Proof. Apply Theorem 3.10 with $P = (X, \tau, \in)$ and $Q = (\tau \setminus \{\emptyset\}, \tau, T)$, where $UTV \leftrightarrow U \cap V \neq \emptyset$. \square

Corollary 3.14 (folklore). *For a topological space X and a point $x \in X$, the games $G_{\mathcal{O}, P}^c(X, x)$ and $G_1(\Omega_x, \Omega_x)$ on X are dual games.*

Proof. Apply Theorem 3.10 with $P = Q = (\tau_x, X, \ni)$. \square

Remark 3.15. *It is worth pointing out that, for two relations $P = (A, B, R)$ and $Q = (C, D, T)$, the games $G(P, Q)$ and $G_1(\text{Dom}(P), \text{Dom}(Q))$ are of interest only if $B \in \text{Dom}(Q)$, for otherwise TWO (resp. ONE) wins every play of $G(P, Q)$ (resp. $G_1(\text{Dom}(P), \text{Dom}(Q))$) trivially. Although we do not include this condition in the definition of these games (since it is not needed for most of our results), the reader should note that, in all of the topological situations we consider in this paper, the sets B and D are the same.*

4. PRODUCTS AND SINGLETON SELECTIONS

In this section we explore the behavior of some selective properties under products, focusing on the game $G(P, Q)$ — and, equivalently, on the game $G_1(\text{Dom}(P), \text{Dom}(Q))$.

The first step towards this goal is to define the product of two relations. Since we are interested in applying the general results to topological properties, we must consider a definition that allows us to do the following in as many situations as possible: If P and Q are relations whose sets of dominating families correspond to a certain topological concept on the topological spaces X and Y respectively, then the set of dominating families in their product $P \otimes Q$ must allow us to describe the same topological concept in the product space $X \times Y$. A natural way of defining such product is the following.

Definition 4.1. *The product relation of two relations $P = (A, B, R)$ and $P' = (A', B', R')$ is defined as $P \otimes P' = (A \times A', B \times B', R \otimes R')$, where*

$$R \otimes R' = \{((a, a'), (b, b')) \in (A \times A') \times (B \times B') : aRb \text{ and } a'R'b'\}.$$

Example 4.2. *Let $P = (\tau \setminus \{\emptyset\}, X, \ni)$ and $Q = (\rho \setminus \{\emptyset\}, Y, \ni)$. Then $P \otimes Q = ((\tau \setminus \{\emptyset\}) \times (\rho \setminus \{\emptyset\}), X \times Y, \ni \otimes \ni)$, where $(U, V) \ni \otimes \ni (x, y) \leftrightarrow (U \ni x \ \& \ V \ni y)$. In this case, we have $\text{Dom}(P) = \mathfrak{D}_X$, $\text{Dom}(Q) = \mathfrak{D}_Y$ and $\text{Dom}(P \otimes Q) = \mathfrak{D}_{X \times Y}$.*

Example 4.3. *Let $P = (X, \tau, \in)$ and $Q = (Y, \rho, \in)$. Then $P \otimes Q = (X \times Y, \tau \times \rho, \in \otimes \in)$, where $(x, y) \in \in \otimes \in (U, V) \leftrightarrow (x \in U \ \& \ y \in V)$. In this case, we have $\text{Dom}(P) = \mathcal{O}_X$ and $\text{Dom}(Q) = \mathcal{O}_Y$. The set $\text{Dom}(P \otimes Q)$ does not correspond exactly to $\mathcal{O}_{X \times Y}$, but rather to the set of covers of the product $X \times Y$ constituted by basic open sets — which, however, is enough to express properties such as “ $X \times Y$ is Lindelöf” and “ $X \times Y$ is Rothberger” as $\text{Dom}(P \otimes Q)$ -Lindelöf and $S_1(\text{Dom}(P \otimes Q), \text{Dom}(P \otimes Q))$ respectively.*

We can now prove the main result of this section — which was described in general terms in the Introduction.

Proposition 4.4. *Let P , P' , Q and Q' be relations. Suppose that ONE has a winning strategy in the game $G(P', Q')$.*

- (a) *If $(\text{Dom}(P), \text{Dom}(Q))$ -Lindelöf holds, then $(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$ -Lindelöf also holds.*
- (b) *If $S_1(\text{Dom}(P), \text{Dom}(Q))$ holds, then $S_1(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$ also holds.*
- (c) *If ONE has a winning strategy in the game $G(P, Q)$, then ONE has a winning strategy in the game $G(P \otimes P', Q \otimes Q')$.*

Proof. Write $P = (A, B, R)$, $P' = (A', B', R')$, $Q = (C, D, T)$ and $Q' = (C', D', T')$, and let $\sigma : {}^{<\omega}B' \rightarrow A'$ be a winning strategy for ONE in the game $G(P', Q')$.

(a) Let $\{(b_i, b'_i) : i \in I\} \in \text{Dom}(P \otimes P')$ be fixed. We will construct indexed families $\langle a'_s : s \in {}^{<\omega}\omega \rangle$ and $\langle i_n^s : s \in {}^{<\omega}\omega, n \in \omega \rangle$ satisfying:

- $a'_s \in A'$ for all $s \in {}^{<\omega}\omega$;
- $i_n^s \in I$ for all $s \in {}^{<\omega}\omega$ and $n \in \omega$;
- $\{b_{i_n^s} : n \in \omega\} \in \text{Dom}(Q)$ for all $s \in {}^{<\omega}\omega$; and
- for each $f \in {}^\omega\omega$,

$$\left(a'_{\emptyset}, b'_{i_{f(0)}}, a'_{f(0)}, b'_{i_{f(0)}}, a'_{f(0), f(1)}, b'_{i_{f(0), f(1)}}, \dots, a'_{f \upharpoonright k}, b'_{i_{f \upharpoonright k}}, \dots \right)$$

is a play of $G(P', Q')$ in which ONE follows the strategy σ .

We proceed by recursion. Suppose that $k \in \omega$ is such that $\langle a'_t : t \in {}^{<k}\omega \rangle$ and $\langle i_n^t : t \in {}^{<k}\omega, n \in \omega \rangle$ have already been constructed, and let $s \in {}^k\omega$ be fixed. Define $a'_s = \sigma \left(\left(b'_{i_{s \upharpoonright j}} \right)_{j < k} \right)$. Note that $\{b_i : i \in I \text{ and } a'_s R' b'_i\} \in \text{Dom}(P)$; hence, by our hypothesis, there is $\{i_n^s : n \in \omega\} \subseteq \{i \in I : a'_s R' b'_i\}$ with $\{b_{i_n^s} : n \in \omega\} \in \text{Dom}(Q)$. This completes the recursion.

We now claim that $\left\{ (b_{i_n^s}, b'_{i_n^s}) : s \in {}^{<\omega}\omega, n \in \omega \right\} \in \text{Dom}(Q \otimes Q')$. Indeed, let $(c, c') \in C \times C'$ be arbitrary. Pick $n_k \in \omega$ recursively for $k \in \omega$ so that $c T b_{i_{n_k}^{(n_j)_{j < k}}}$. Now

$$\left(a'_{\emptyset}, b'_{i_{n_0}}, a'_{(n_0)}, b'_{i_{n_1}^{(n_0)}}, a'_{(n_0, n_1)}, b'_{i_{n_2}^{(n_0, n_1)}}, \dots, a'_{(n_j)_{j < k}}, b'_{i_{n_k}^{(n_j)_{j < k}}}, \dots \right)$$

is a play $G(P', Q')$ in which ONE makes use of the winning strategy σ , so there is $k \in \omega$ such that $c' T' b'_{i_{n_k}^{(n_j)_{j < k}}}$. Hence $(c, c') T \otimes T' \left(b_{i_{n_k}^{(n_j)_{j < k}}}, b'_{i_{n_k}^{(n_j)_{j < k}}} \right)$, as required.

(b) Let $(Z_n)_{n \in \omega}$ be a sequence of elements of $\text{Dom}(P \otimes P')$, and write $Z_n = \{(b_i^n, b'_i^n) : i \in I_n\}$ for each $n \in \omega$. Now write $\omega = \{m_k^s : s \in {}^{<\omega}\omega, k \in \omega\}$ with $m_k^s = m_l^t$ only if $(s, k) = (t, l)$. We will assign to each $s \in {}^{<\omega}\omega$ and each $k \in \omega$ an $i_k^s \in I_{m_k^s}$ according to the following procedure.

Suppose that $h \in \omega$ is such that $i_k^t \in I_{m_k^t}$ has already been chosen for all $t \in {}^{<h}\omega$ and all $k \in \omega$. Now let $s \in {}^h\omega$ be fixed. Define $a'_s = \sigma \left(\left(b'_{i_{s \upharpoonright j}^{m_{s \upharpoonright j}^s}} \right)_{j < h} \right)$.

Since $\{b_i^{m_k^s} : i \in I_{m_k^s} \text{ and } a'_s R' b_i^{m_k^s}\} \in \text{Dom}(P)$ for each $k \in \omega$, we may apply $S_1(\text{Dom}(P), \text{Dom}(Q))$ to obtain a sequence $(i_k^s)_{k \in \omega}$ satisfying:

- $i_k^s \in I_{m_k^s}$ for each $k \in \omega$;
- $a'_s R' b_{i_k^s}^{m_k^s}$ for each $k \in \omega$; and
- $\{b_{i_k^s}^{m_k^s} : k \in \omega\} \in \text{Dom}(Q)$.

By recursion, this concludes the definition of i_k^s for $s \in {}^{<\omega}\omega$ and $k \in \omega$. (In each step, what we have done is: if $s = (k_0, k_1, \dots, k_{h-1}) \in {}^h\omega$, then a'_s is ONE's move in a play of $G(P', Q')$ whose history so far is

$$\left(a'_{\emptyset}, b_{i_{\emptyset}^{\emptyset}}^{m_{\emptyset}^{\emptyset}}, a'_{(k_0)}, b_{i_{k_0}^{(k_0)}}^{m_{k_0}^{(k_0)}}, a'_{(k_0, k_1)}, b_{i_{(k_0, k_1)}^{(k_0, k_1)}}^{m_{(k_0, k_1)}^{(k_0, k_1)}}, \dots, a'_{(k_0, k_1, \dots, k_{h-2})}, b_{i_{(k_0, k_1, \dots, k_{h-2})}^{(k_0, k_1, \dots, k_{h-2})}}^{m_{(k_0, k_1, \dots, k_{h-2})}^{(k_0, k_1, \dots, k_{h-2})}} \right);$$

then, in view of the fact that $Z_n(a') = \{b_i^n : i \in I_n \text{ and } a' R' b_i^n\}$ is an element of $\text{Dom}(P)$ for all $a' \in A'$ and $n \in \omega$, we have made use of $S_1(\text{Dom}(P), \text{Dom}(P'))$ to select from each $I_{m_k^s}$ with $k \in \omega$ an element $b_{i_k^s}^{m_k^s}$ of $Z_{m_k^s}(a'_s)$ in such a way that $\{b_{i_k^s}^{m_k^s} : k \in \omega\} \in \text{Dom}(Q)$.

We now claim that $\{(b_{i_k^s}^{m_k^s}, b_{i_k^s}^{m_k^s}) : s \in {}^{<\omega}\omega, k \in \omega\} \in \text{Dom}(Q \otimes Q')$. Indeed, let $(c, c') \in C \times C'$ be arbitrary. Since $\{b_{i_k^s}^{m_k^s} : k \in \omega\} \in \text{Dom}(Q)$ for every $s \in {}^{<\omega}\omega$,

we may recursively pick, for each $r \in \omega$, a $k_r \in \omega$ such that $c T b_{i_{k_r}}^{m_{k_r}^{(k_j)_{j < r}}}$. Then

$$\left(a'_{\emptyset}, b_{i_{\emptyset}^{\emptyset}}^{m_{\emptyset}^{\emptyset}}, a'_{(k_0)}, b_{i_{k_0}^{(k_0)}}^{m_{k_0}^{(k_0)}}, a'_{(k_0, k_1)}, b_{i_{(k_0, k_1)}^{(k_0, k_1)}}^{m_{(k_0, k_1)}^{(k_0, k_1)}}, \dots, a'_{(k_j)_{j < r}}, b_{i_{(k_j)_{j < r}}^{(k_j)_{j < r}}}^{m_{(k_j)_{j < r}}^{(k_j)_{j < r}}}, \dots \right)$$

is a play of $G(P', Q')$ in which ONE follows the winning strategy σ , whence there is $r \in \omega$ with $c' T' b_{i_{k_r}}^{m_{k_r}^{(k_j)_{j < r}}}$. Therefore, $(c, c') T \otimes T' \left(b_{i_{k_r}}^{m_{k_r}^{(k_j)_{j < r}}}, b_{i_{k_r}}^{m_{k_r}^{(k_j)_{j < r}}} \right)$.

(c) Let $\varphi : {}^{<\omega}B \rightarrow A$ be a winning strategy for ONE in $G(P, Q)$. Consider a partition $\omega = \bigcup \{L_s : s \in {}^{<\omega}\omega\}$, where each L_s is infinite and $\min(L_s) > \max(\text{im}(s))$ for all $s \in {}^{<\omega}\omega \setminus \{\emptyset\}$. For each $s \in {}^{<\omega}\omega$, write $L_s = \{m_k^s : k \in \omega\}$ with $m_k^s < m_{k+1}^s$ for all $k \in \omega$. A strategy for ONE in the game $G(P \otimes P', Q \otimes Q')$ will then be defined as follows.

Suppose that, at the inning $n \in \omega$ of a play of $G(P \otimes P', Q \otimes Q')$, the sequence of moves so far is

$$((a_0, a'_0), (b_0, b'_0), (a_1, a'_1), (b_1, b'_1), \dots, (a_{n-1}, a'_{n-1}), (b_{n-1}, b'_{n-1})).$$

Let $s \in {}^{<\omega}\omega$ and $k \in \omega$ be such that $n = m_k^s \in L_s$. Then ONE's move in the current inning is $(a_n, a'_n) \in A \times A'$, where $a_n = \varphi((b_{m_i^s})_{i < k})$ and $a'_n = \sigma((b'_{s(j)})_{j < \text{dom}(s)})$. That is: ONE follows the strategy φ in a play of $G(P, Q)$ whose previous moves by TWO are not all of b_0, b_1, \dots, b_{n-1} , but only those with indices in the set $L_s \cap n$; in a similar fashion, ONE makes use of the strategy σ in a play of $G(P', Q')$ whose previous moves by TWO are not all of $b'_0, b'_1, \dots, b'_{n-1}$, but only those with indices listed by the sequence s .

We claim that this is a winning strategy for ONE in $G(P \otimes P', Q \otimes Q')$. Indeed, suppose that

$$((a_0, a'_0), (b_0, b'_0), (a_1, a'_1), (b_1, b'_1), \dots, (a_n, a'_n), (b_n, b'_n), \dots)$$

is a play of $G(P \otimes P', Q \otimes Q')$ in which ONE adopts the strategy above described, and let $(c, c') \in C \times C'$ be arbitrary. We can construct a strictly increasing infinite sequence $(n_j)_{j \in \omega} \in {}^\omega \omega$ by picking recursively $n_j \in L_{(n_i)_{i < j}}$ with cTb_{n_j} for each $j \in \omega$ — such an n_j must exist, for otherwise

$$(a_{m_0}^{(n_i)_{i < j}}, b_{m_0}^{(n_i)_{i < j}}, a_{m_1}^{(n_i)_{i < j}}, b_{m_1}^{(n_i)_{i < j}}, \dots, a_{m_k}^{(n_i)_{i < j}}, b_{m_k}^{(n_i)_{i < j}}, \dots)$$

would be a play of $G(P, Q)$ in which ONE plays according to the winning strategy φ and loses. Now

$$(a'_{n_0}, b'_{n_0}, a'_{n_1}, b'_{n_1}, \dots, a'_{n_j}, b'_{n_j}, \dots)$$

is a play of $G(P', Q')$ in which ONE employs the winning strategy σ , whence $c'T'b'_{n_k}$ must hold for some $k \in \omega$. Thus $(c, c')T \otimes T'(b_{n_k}, b'_{n_k})$, as required. \square

In view of Theorem 3.10, Proposition 4.4 can be restated as follows.

Proposition 4.5. *Let P , P' , Q and Q' be relations. Suppose that TWO has a winning strategy in the game $G_1(\text{Dom}(P'), \text{Dom}(Q'))$.*

- (a) *If $(\text{Dom}(P), \text{Dom}(Q))$ -Lindelöf holds, then $(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$ -Lindelöf also holds.*
- (b) *If $S_1(\text{Dom}(P), \text{Dom}(Q))$ holds, then $S_1(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$ also holds.*
- (c) *If TWO has a winning strategy in the game $G_1(\text{Dom}(P), \text{Dom}(Q))$, then TWO has a winning strategy in the game $G_1(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$.*

We now present some instances of the previous propositions. The following list of results is not meant to exhaust the consequences that can be obtained from Propositions 4.4 and 4.5, but rather to illustrate some of the contexts to which they can be applied.

Corollary 4.6. *If TWO has a winning strategy in the Rothberger game on a topological space X , then X is productively Rothberger.*

Proof. Let Y be a Rothberger space. Now apply Proposition 4.5(b) with $P' = Q' = (X, \tau, \in)$ and $P = Q = (Y, \rho, \in)$. \square

It is worth comparing Corollary 4.6 with Theorem 11(3) of [9], in which a product of two metric spaces is proven to be Rothberger under a weaker hypothesis on one of the spaces and a stronger hypothesis on the other.

The next result was first stated in [37]; see also Theorem 3.1 of [43].²

Corollary 4.7 (Telgársky [37]). *The property “ONE has a winning strategy in the point-open game” (equivalently, “TWO has a winning strategy in the Rothberger game”) is preserved under finite products.*

Proof. Let X and Y be spaces on which ONE has a winning strategy in the point-open game. Apply Proposition 4.4(c) with $P = Q = (X, \tau, \in)$ and $P' = Q' = (Y, \rho, \in)$. \square

²We thank Piotr Szewczak for bringing the paper [43] to our attention.

The next result is also a consequence of Theorem 3.1 of [43]. By the *compact-open game* on a topological space X we mean the following game: In each inning $n \in \omega$, ONE chooses a compact subset C_n of X , and then TWO picks an open set U_n with $C_n \subseteq U_n$; ONE wins if $X = \bigcup_{n \in \omega} U_n$, and loses otherwise.

Corollary 4.8 (Yajima [43]). *The property “ONE has a winning strategy in the compact-open game” is preserved under finite products.*

Proof. Let X and Y be spaces on which ONE has a winning strategy in the compact-open game. Apply Proposition 4.4(c) with $P = (K(X), \tau, \subseteq)$, $Q = (X, \tau, \in)$, $P' = (K(Y), \rho, \subseteq)$ and $Q' = (Y, \rho, \in)$. The result follows from the observation that, whenever $C_1 \in K(X)$, $C_2 \in K(Y)$ and W is an open subset of $X \times Y$ with $C_1 \times C_2 \subseteq W$, there exist $U_1 \in \tau$ and $U_2 \in \rho$ satisfying $C_1 \times C_2 \subseteq U_1 \times U_2 \subseteq W$. \square

Corollary 4.9. *Let X be a topological space on which TWO has a winning strategy in the game $G_1(\Omega, \Omega)$. Then:*

- (a) *the product $X \times Y$ satisfies $S_1(\Omega, \Omega)$ for every topological space Y satisfying $S_1(\Omega, \Omega)$;*
- (a') *if Y is a topological space that is Rothberger in every finite power, then the product $X \times Y$ is Rothberger in every finite power;*
- (b) *if TWO has a winning strategy in the game $G_1(\Omega, \Omega)$ on a topological space Y , then TWO has a winning strategy in $G_1(\Omega, \Omega)$ on the product $X \times Y$.*

Proof. By applying Proposition 4.5(b)-(c) with $P' = Q' = ([X]^{<\aleph_0}, \tau, \subseteq)$ and $P = Q = ([Y]^{<\aleph_0}, \rho, \subseteq)$, we obtain (a) and (b) — note that every ω -cover \mathcal{U} of the product $X \times Y$ has an open refinement \mathcal{V} that is also an ω -cover and such that every element of \mathcal{V} is a basic open set of the form $U \times V$. Now (a') follows from the fact that a topological space satisfies $S_1(\Omega, \Omega)$ if and only if all of its finite powers are Rothberger [31]. \square

For the next result, recall that a space X is *weakly Lindelöf* [17] if $(\mathcal{O}_X, \mathcal{D}_X)$ -Lindelöf holds. Furthermore, X is *weakly Rothberger* [14] if $S_1(\mathcal{O}_X, \mathcal{D}_X)$ holds.

Corollary 4.10. *Let X be a topological space on which TWO has a winning strategy in the game $G_1(\mathcal{O}, \mathcal{D})$. Then:*

- (a) *the topological product $X \times Y$ is weakly Lindelöf whenever Y is a weakly Lindelöf space;*
- (b) *the topological product $X \times Y$ is weakly Rothberger whenever Y is a weakly Rothberger space;*
- (c) *TWO has a winning strategy in the game $G_1(\mathcal{O}, \mathcal{D})$ on the product $X \times Y$ whenever TWO has a winning strategy in $G_1(\mathcal{O}, \mathcal{D})$ on the topological space Y .³*

Proof. Apply Proposition 4.5 with $P' = (X, \tau, \in)$, $Q' = (\tau \setminus \{\emptyset\}, \tau, T)$, $P = (Y, \rho, \in)$ and $Q = (\rho \setminus \{\emptyset\}, \rho, T')$, where $T = \{(U, V) \in (\tau \setminus \{\emptyset\}) \times \tau : U \cap V \neq \emptyset\}$ and $T' = \{(U, V) \in (\rho \setminus \{\emptyset\}) \times \rho : U \cap V \neq \emptyset\}$. \square

The next corollary gives us an application to a nontopological context — namely, variations on the countable chain condition for partial orders (see e.g. [26, Chapter III] for further details on the concepts involved in this result).

³A result similar to Corollary 4.10(c) was independently obtained in [6] under a stronger assumption.

Corollary 4.11. *Let \mathbb{P} be a partial order and $\mathcal{PD}_{\mathbb{P}} = \{W \subseteq \mathbb{P} : W \text{ is predense in } \mathbb{P}\}$. Suppose that TWO has a winning strategy in the game $G_1(\mathcal{PD}_{\mathbb{P}}, \mathcal{PD}_{\mathbb{P}})$. Then:*

- (a) *the partial order $\mathbb{P} \times \mathbb{Q}$ is c.c.c. for every c.c.c. partial order \mathbb{Q} ;*
- (b) *the partial order $\mathbb{P} \times \mathbb{Q}$ satisfies $S_1(\mathcal{PD}, \mathcal{PD})$ for every partial order \mathbb{Q} satisfying $S_1(\mathcal{PD}, \mathcal{PD})$.*

Proof. Apply Proposition 4.5(a)-(b) with $P' = Q' = (\mathbb{P}, \mathbb{P}, \perp)$ and $P = Q = (\mathbb{Q}, \mathbb{Q}, \perp)$. The result follows from the observation that a partial order is c.c.c. if and only if every element of \mathcal{PD} has a countable subset that is also an element of \mathcal{PD} — since for every $X \in \mathcal{PD}$ there is a maximal antichain $A \in \mathcal{PD}$ such that every element of X has an extension in A . \square

It is worth mentioning that Corollary 1.7 of [15] is equivalent — by Theorem 4.1 of the same paper — to the statement that the property “TWO has a winning strategy in $G_1(\mathcal{PD}, \mathcal{PD})$ ” is preserved under arbitrary products with finite support.

The next result may be viewed as a topological counterpart of Corollary 4.11.

Corollary 4.12. *Let X be a topological space on which TWO has a winning strategy in the game $G_1(\mathcal{D}, \mathcal{D})$. Then:*

- (a) *the product $X \times Y$ is c.c.c. for every c.c.c. topological space Y ;*
- (b) *the product $X \times Y$ satisfies $S_1(\mathcal{D}, \mathcal{D})$ for every topological space Y that satisfies $S_1(\mathcal{D}, \mathcal{D})$.*

Proof. Apply Corollary 4.11 with $\mathbb{P} = (\tau \setminus \{\emptyset\}, \subseteq)$ and $\mathbb{Q} = (\rho \setminus \{\emptyset\}, \subseteq)$. \square

Again by Corollary 1.7 of [15] (see comment after Corollary 4.11), the property “TWO has a winning strategy in $G_1(\mathcal{D}, \mathcal{D})$ ” is preserved in arbitrary (Tychonoff) products of topological spaces.

Note that, as a consequence of Corollary 4.12(a), TWO does not have a winning strategy in the game $G_1(\mathcal{D}, \mathcal{D})$ played on a Suslin line; therefore, the game $G_1(\mathcal{D}, \mathcal{D})$ is undetermined on Suslin lines — see [32, Theorem 14 and observation following Problem 1].

For the next result, we recall that a topological space is *R-separable* [33] if it satisfies $S_1(\mathfrak{D}, \mathfrak{D})$. For a topological space X , we define $\delta(X) = \sup\{d(Z) : Z \in \mathfrak{D}_X\}$, where $d(Z) = \min\{|D| : D \in \mathfrak{D}_Z\} + \aleph_0$.

Corollary 4.13. *Let X be a topological space such that TWO has a winning strategy in the game $G_1(\mathfrak{D}, \mathfrak{D})$ on X . Then:*

- (a) *$\delta(X \times Y) = \aleph_0$ for every topological space Y with $\delta(Y) = \aleph_0$;*
- (b) *$X \times Y$ is R-separable for every R-separable space Y ;*
- (c) *TWO has a winning strategy in the game $G_1(\mathfrak{D}, \mathfrak{D})$ on the product $X \times Y$ for every topological space Y on which TWO has a winning strategy in the game $G_1(\mathfrak{D}, \mathfrak{D})$.*

Proof. Apply Proposition 4.5 with $P' = Q' = (\tau \setminus \{\emptyset\}, X, \ni)$ and $P = Q = (\rho \setminus \{\emptyset\}, Y, \ni)$. \square

For our last result in this section, recall that a topological space X is *countably tight* at a point $x \in X$ if Ω_x -Lindelöf holds on X , and has *countable strong fan tightness* at x [31] if $S_1(\Omega_x, \Omega_x)$ holds on X .

Corollary 4.14. *Let X be a topological space and $x \in X$. Suppose that TWO has a winning strategy in the game $G_1(\Omega_x, \Omega_x)$ on X . Then:*

- (a) if a topological space Y is countably tight at a point $y \in Y$, then $X \times Y$ is countably tight at (x, y) ;
- (b) if a topological space Y has countable strong fan tightness at $y \in Y$, then the product space $X \times Y$ has countable strong fan tightness at the point (x, y) ;
- (c) if a topological space Y is such that TWO has a winning strategy in the game $G_1(\Omega_y, \Omega_y)$ for a point $y \in Y$, then TWO has a winning strategy in the game $G_1(\Omega_{(x,y)}, \Omega_{(x,y)})$ on the product $X \times Y$.

Proof. Apply Proposition 4.5(b) with $P' = Q' = (\{U \in \tau : x \in U\}, X, \ni)$ and $P = Q = (\{V \in \rho : y \in V\}, Y, \ni)$. \square

We note that Corollary 4.14(a) extends Corollary 2.4 of [4], in which the same conclusion is obtained under the assumption that X is completely regular. This result will be further improved in Corollary 7.6, in view of the fact that, if $\pi\mathcal{N}_x^{\aleph_0}$ -Lindelöf holds on X , then X is productively countably tight at x [11, Corollary 2.3].

5. PRODUCTS AND FINITE SELECTIONS

By Corollary 3 of [38], Corollary 4.8 is equivalent to the statement that the property “TWO has a winning strategy in the Menger game” is finitely productive in the realm of regular spaces. We will now give a direct proof⁴ of this fact without assuming any separation axioms, which will serve as motivation for Proposition 5.3.

Proposition 5.1. *Let X and Y be topological spaces such that TWO has a winning strategy in both of the games $G_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ and $G_{\text{fin}}(\mathcal{O}_Y, \mathcal{O}_Y)$. Then TWO has a winning strategy in the game $G_{\text{fin}}(\mathcal{O}_{X \times Y}, \mathcal{O}_{X \times Y})$.*

Proof. Let $\varphi : {}^{<\omega}\mathcal{O}_X \setminus \{\emptyset\} \rightarrow [\tau]^{<\aleph_0} \setminus \{\emptyset\}$ and $\sigma : {}^{<\omega}\mathcal{O}_Y \setminus \{\emptyset\} \rightarrow [\rho]^{<\aleph_0} \setminus \{\emptyset\}$ be winning strategies for TWO in $G_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ and $G_{\text{fin}}(\mathcal{O}_Y, \mathcal{O}_Y)$ respectively. Let $\mathcal{P} = \bigcup_{n \in \omega} ({}^n\omega \times {}^n\omega)$, and write $\omega = \bigcup \{L_s^t : (s, t) \in \mathcal{P}\}$ with

- $|L_s^t| = \aleph_0$ for every $(s, t) \in \mathcal{P}$; and
- $\min L_s^t > \max(\text{im}(s))$ for every $(s, t) \in \mathcal{P} \setminus \{(\emptyset, \emptyset)\}$.

Enumerate each set L_s^t as $L_s^t = \{l_s^t(j) : j \in \omega\}$, where $l_s^t(j) < l_s^t(j+1)$ for all $j \in \omega$.

In order to construct a winning strategy for TWO in the game $G_{\text{fin}}(\mathcal{O}_{X \times Y}, \mathcal{O}_{X \times Y})$, we may assume that, in each inning $n \in \omega$ of this game, ONE plays an open cover \mathcal{U}_n of $X \times Y$ of the form $\mathcal{U}_n = \{U_i^n \times V_i^n : i \in I_n\}$. Given such an open cover, for each $x \in X$, define $I_n^x = \{i \in I_n : x \in U_i^n\}$ and $\mathcal{V}_n^x = \{V_i^n : i \in I_n^x\} \in \mathcal{O}_Y$. In each inning $n \in \omega$, we will make use of the strategy σ to assign to each $x \in X$ a finite nonempty subset of \mathcal{V}_n^x — that is, a finite nonempty subset F_n^x of I_n^x . We will then define $W_n^x = \bigcap_{i \in F_n^x} U_i^n$ for each $x \in X$, and then make use of the strategy φ to choose finitely many elements of the open cover $\mathcal{W}_n = \{W_n^x : x \in X\}$ of X — that is, a finite subset $\{x_k^n : k \in \omega\}$ of X , here enumerated with infinite repetition of all of the terms. TWO’s answer to \mathcal{U}_n will then be the finite set $\mathcal{F}_n = \bigcup_{k \in \omega} \{U_i^n \times V_i^n : i \in F_n^{x_k^n}\}$. In order to make sure that this will define a winning strategy, when making use of σ we will consider not all of the previous innings, but only those listed (in a sense that will become clear in the next paragraph) by the sequences $s, t \in {}^{<\omega}\omega$ such that $n \in L_s^t$; also, when making use of

⁴We could put this proof together thanks to an idea due to Leandro Aurichi, to whom we are very grateful.

φ we will consider a play of $\mathbf{G}_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ whose history is given not by all of the innings $0, 1, \dots, n-1$, but only those in $L_s^t \cap n$.

Now for the details of the procedure. Suppose that, in the inning $n \in \omega$ of $\mathbf{G}_{\text{fin}}(\mathcal{O}_{X \times Y}, \mathcal{O}_{X \times Y})$, the play so far is $(\mathcal{U}_0, \mathcal{F}_0, \mathcal{U}_1, \mathcal{F}_1, \dots, \mathcal{U}_n)$, where each \mathcal{F}_m is of the form $\mathcal{F}_m = \bigcup_{k \in \omega} \{U_i^m \times V_i^m : i \in F_m^{x_k^m}\}$, as described in the previous paragraph. Let $(s, t) \in \mathcal{P}$ and $j \in \omega$ be such that $n = l_s^t(j) \in L_s^t$. For each $x \in X$, let

$$F_n^x \in [I_n^x]^{<\aleph_0} \setminus \{\emptyset\} \text{ be such that } \sigma \left(\left(\mathcal{V}_{s(r)}^{x_{t(r)}^{s(r)}} \right)_{r \in \text{dom}(s)} \frown (\mathcal{V}_n^x) \right) = \{V_i^n : i \in F_n^x\}.$$

We can now define the open neighborhood $W_n^x = \bigcap_{i \in F_n^x} U_i^n$ for each $x \in X$, and then consider $\mathcal{W}_n = \{W_n^x : x \in X\} \in \mathcal{O}_X$. Now let $\{x_k^n : k \in \omega\}$ be a finite subset of X , enumerated with infinite repetition of all of the terms, satisfying $\varphi \left((\mathcal{W}_{l_s^t(h)})_{h \leq j} \right) = \{W_n^{x_k^n} : k \in \omega\}$. TWO's answer in the n -th inning of the play $(\mathcal{U}_0, \mathcal{F}_0, \mathcal{U}_1, \mathcal{F}_1, \dots, \mathcal{U}_n)$ of $\mathbf{G}_{\text{fin}}(\mathcal{O}_{X \times Y}, \mathcal{O}_{X \times Y})$ is then the finite subset $\mathcal{F}_n = \bigcup_{k \in \omega} \{U_i^n \times V_i^n : i \in F_n^{x_k^n}\}$ of \mathcal{U}_n .

Let us now prove that this defines a winning strategy for TWO in $\mathbf{G}_{\text{fin}}(\mathcal{O}_{X \times Y}, \mathcal{O}_{X \times Y})$.

Let $(x, y) \in X \times Y$ be arbitrary. Pick $n_0 \in L_\emptyset^\emptyset$ with $x \in \bigcup_{k \in \omega} W_{n_0}^{x_{k_0}^{n_0}}$ — such an n_0 must exist, for otherwise

ONE	TWO
$\mathcal{W}_{l_\emptyset^\emptyset(0)}$	$\left\{ W_{l_\emptyset^\emptyset(0)}^{x_k^{l_\emptyset^\emptyset(0)}} : k \in \omega \right\}$
$\mathcal{W}_{l_\emptyset^\emptyset(1)}$	$\left\{ W_{l_\emptyset^\emptyset(1)}^{x_k^{l_\emptyset^\emptyset(1)}} : k \in \omega \right\}$
\vdots	\vdots
$\mathcal{W}_{l_\emptyset^\emptyset(j)}$	$\left\{ W_{l_\emptyset^\emptyset(j)}^{x_k^{l_\emptyset^\emptyset(j)}} : k \in \omega \right\}$
\vdots	\vdots

would be a play of $\mathbf{G}_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ in which TWO plays according to the winning strategy φ and loses. Let then $k_0 \in \omega$ be such that $x \in W_{n_0}^{x_{k_0}^{n_0}}$. Similarly, we can recursively pick $n_r \in L_{(n_l)_{l < r}}^{(k_l)_{l < r}}$ and $k_r \in \omega$ satisfying $x \in W_{n_r}^{x_{k_r}^{n_r}}$ for each $r \in \omega$, thus defining the sequences $(n_r)_{r \in \omega} \in {}^\omega \omega$ and $(k_r)_{r \in \omega} \in {}^\omega \omega$. Now

ONE	TWO
$\mathcal{V}_{n_0}^{x_{k_0}^{n_0}}$	$\left\{ V_i^{n_0} : i \in F_{n_0}^{x_{k_0}^{n_0}} \right\}$
$\mathcal{V}_{n_1}^{x_{k_1}^{n_1}}$	$\left\{ V_i^{n_1} : i \in F_{n_1}^{x_{k_1}^{n_1}} \right\}$
\vdots	\vdots
$\mathcal{V}_{n_r}^{x_{k_r}^{n_r}}$	$\left\{ V_i^{n_r} : i \in F_{n_r}^{x_{k_r}^{n_r}} \right\}$
\vdots	\vdots

is a play of $\mathbf{G}_{\text{fin}}(\mathcal{O}_Y, \mathcal{O}_Y)$ in which TWO follows the winning strategy σ ; therefore, there is $h \in \omega$ such that, for some $i' \in F_{n_r}^{x_{k_r}^{n_r}}$, we have $y \in V_{i'}^{n_r}$. Since $x \in W_{n_r}^{x_{k_r}^{n_r}} = \bigcap_{i \in F_{n_r}^{x_{k_r}^{n_r}}} U_i^{n_r} \subseteq U_{i'}^{n_r}$ also holds, it follows that $(x, y) \in U_{i'}^{n_r} \times V_{i'}^{n_r}$. Thus, the sets played by TWO cover the product $X \times Y$. \square

In order to capture the main aspect needed to adapt the proof of Proposition 5.1 to the general setting of relations, we define the following auxiliary concept.

Definition 5.2. Let $P = (A, B, R)$ be a relation and \preceq be a partial order on the set B . We say that \preceq is

- downwards P -compatible if, for all $a \in A$ and $b_1, b_2 \in B$,
$$(aRb_1 \ \& \ aRb_2) \rightarrow \exists \tilde{b} \in B \ (aR\tilde{b} \ \& \ \tilde{b} \preceq b_1 \ \& \ \tilde{b} \preceq b_2);$$
- upwards P -compatible if, for all $a \in A$ and $b_1, b_2 \in B$,
$$(aRb_1 \ \& \ b_1 \preceq b_2) \rightarrow aRb_2.$$

We can now state the main result of this section.

Proposition 5.3. Let $P = (A, B, R)$, $P' = (A', B', R')$, $Q = (C, D, T)$ and $Q' = (C', D', T')$ be relations with $B = D$ such that there is a partial order \preceq on B that is both downwards P -compatible and upwards Q -compatible. Suppose that TWO has a winning strategy in the game $\mathbf{G}_{\text{fin}}(\text{Dom}(P'), \text{Dom}(Q'))$.

- (a) If $(\text{Dom}(P), \text{Dom}(Q))$ -Lindelöf holds, then $(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$ -Lindelöf holds.
- (b) If $\mathbf{S}_{\text{fin}}(\text{Dom}(P), \text{Dom}(Q))$ holds, then $\mathbf{S}_{\text{fin}}(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$ also holds.
- (c) If TWO has a winning strategy in the game $\mathbf{G}_{\text{fin}}(\text{Dom}(P), \text{Dom}(Q))$, then TWO has a winning strategy in the game $\mathbf{G}_{\text{fin}}(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$.

Proof. Let $\sigma : {}^{<\omega}\text{Dom}(P') \setminus \{\emptyset\} \rightarrow [B']^{<\aleph_0} \setminus \{\emptyset\}$ be a winning strategy for TWO in the game $\mathbf{G}_{\text{fin}}(\text{Dom}(P'), \text{Dom}(Q'))$.

- (a) Let $\{(b_i, b'_i) : i \in I\} \in \text{Dom}(P \otimes P')$ be fixed. For each $x \in A$, define $Z_x = \{b'_i : i \in I_x\}$, where $I_x = \{i \in I : xRb_i\}$; note that $Z_x \in \text{Dom}(P')$.

We will construct indexed families $\langle F_x^s : s \in {}^{<\omega}\omega, x \in A \rangle$ and $\langle a_n^s : s \in {}^{<\omega}\omega, n \in \omega \rangle$ with

- $F_x^s \in [I_x]^{<\aleph_0} \setminus \{\emptyset\}$ for all $s \in {}^{<\omega}\omega$ and $x \in A$; and
- $a_n^s \in A$ for all $s \in {}^{<\omega}\omega$ and $n \in \omega$.

We proceed recursively as follows.

Suppose that $k \in \omega$ is such that $\langle a_n^t : t \in {}^{<k}\omega, n \in \omega \rangle$ and $\langle F_x^t : t \in {}^{<k}\omega, x \in A \rangle$ have already been constructed, and let $s \in {}^k\omega$ be fixed. For each $x \in A$, let $F_x^s \in [I_x]^{<\aleph_0} \setminus \{\emptyset\}$ be such that

$$\sigma \left(\left(Z_{a_{s(j)}^s} \right)_{j < k} \smallfrown (Z_x) \right) = \{b'_i : i \in F_x^s\}.$$

Since \preceq is downwards P -compatible, for each $x \in A$ we may then choose $b_x^s \in B$ satisfying $x R b_x^s$ and $b_x^s \preceq b_i$ for all $i \in F_x^s$, thus obtaining a set $\{b_x^s : x \in A\} \in \text{Dom}(P)$. By our hypothesis, this set has a countable subset that is an element of $\text{Dom}(Q)$; let then $(a_n^s)_{n \in \omega}$ be a sequence of elements of A with $\{b_{a_n^s}^s : n \in \omega\} \in \text{Dom}(Q)$. This concludes the recursive construction.

The proof will be finished once we show that

$$\bigcup_{s \in {}^{<\omega}\omega} \bigcup_{n \in \omega} \{(b_i, b'_i) : i \in F_{a_n^s}^s\} \in \text{Dom}(Q \otimes Q').$$

To this end, let $(c, c') \in C \times C'$ be arbitrary. Pick recursively, for each $k \in \omega$, an $n_k \in \omega$ such that $c T b_{a_{n_k}}^{(n_j)_{j < k}} \smallfrown$ — this is possible since $\left\{ b_{a_n}^{(n_j)_{j < k}} : n \in \omega \right\} \in \text{Dom}(Q)$. It follows that

ONE	TWO
$Z_{a_{n_0}^\emptyset}$	$\left\{ b'_i : i \in F_{a_{n_0}^\emptyset}^\emptyset \right\}$
$Z_{a_{n_1}^{(n_0)}}$	$\left\{ b'_i : i \in F_{a_{n_1}^{(n_0)}}^{(n_0)} \right\}$
$Z_{a_{n_2}^{(n_0, n_1)}}$	$\left\{ b'_i : i \in F_{a_{n_2}^{(n_0, n_1)}}^{(n_0, n_1)} \right\}$
\vdots	\vdots
$Z_{a_{n_k}^{(n_j)_{j < k}}}$	$\left\{ b'_i : i \in F_{a_{n_k}^{(n_j)_{j < k}}}^{(n_j)_{j < k}} \right\}$
\vdots	\vdots

is a play of $\mathbf{G}_{\text{fin}}(\text{Dom}(P'), \text{Dom}(Q'))$ in which TWO makes use of the winning strategy σ ; hence, for some $k \in \omega$, there is $i \in F_{a_{n_k}}^{(n_j)_{j < k}}$ such that $c' T' b'_i$. Since \preceq is upwards Q -compatible and $c T b_{a_{n_k}}^{(n_j)_{j < k}}$, we also have $c T b_i$; thus, $(c, c') T \otimes T' (b_i, b'_i)$, as required.

(b) Let $(Z_n)_{n \in \omega}$ be a sequence of elements of $\text{Dom}(P \otimes P')$. For each $n \in \omega$, write $Z_n = \{(b_i^n, b_i'^n) : i \in I_n\}$ and, for each $x \in A$, define $I_n^x = \{i \in I_n : xRb_i^n\}$ and $Z_n^x = \{b_i'^n : i \in I_n^x\} \in \text{Dom}(P')$.

Let $\mathcal{P} = \bigcup_{j \in \omega} ({}^j\omega \times {}^j\omega)$. Fix a partition $\omega = \bigcup_{(s,t) \in \mathcal{P}} L_{s,t}$ with $|L_{s,t}| = \aleph_0$ for all $(s,t) \in \mathcal{P}$, and write $L_{s,t} = \{l_m^{s,t} : m \in \omega\}$ with $l_{m_1}^{s,t} \neq l_{m_2}^{s,t}$ if $m_1 \neq m_2$. We will construct an indexed family $\langle a_{s,t} : (s,t) \in \mathcal{P} \setminus \{(\emptyset, \emptyset)\} \rangle$ of elements of A as follows.

Suppose that $h \in \omega$ is such that $a_{s,t}$ has already been defined for all $(s,t) \in \bigcup_{0 < j \leq h} ({}^j\omega \times {}^j\omega)$, and let $(\tilde{s}, \tilde{t}) \in {}^h\omega \times {}^h\omega$ be fixed. For each $x \in A$ and each $m \in \omega$, let $F_m^{\tilde{s}, \tilde{t}}(x) \in \left[l_{\tilde{s}, \tilde{t}}^x \right]^{< \aleph_0} \setminus \{\emptyset\}$ be such that

$$\sigma \left(\left(Z_{l_{\tilde{s}, \tilde{t}}^{a_{\tilde{s}, \tilde{t}}(k+1), \tilde{t}(k+1)}} \right)_{k < h} \cap \left(Z_{l_{\tilde{s}, \tilde{t}}^x} \right) \right) = \left\{ l_i^{\tilde{s}, \tilde{t}} : i \in F_m^{\tilde{s}, \tilde{t}}(x) \right\};$$

then, making use of the fact that \preceq is downwards P -compatible, pick $b_m^{\tilde{s}, \tilde{t}}(x) \in B$ satisfying $xRb_m^{\tilde{s}, \tilde{t}}(x)$ and $b_m^{\tilde{s}, \tilde{t}}(x) \preceq l_i^{\tilde{s}, \tilde{t}}$ for all $i \in F_m^{\tilde{s}, \tilde{t}}(x)$. Since $\{b_m^{\tilde{s}, \tilde{t}}(x) : x \in A\} \in \text{Dom}(P)$ for each $m \in \omega$, we can apply $S_{\text{fin}}(\text{Dom}(P), \text{Dom}(Q))$ and thus obtain a sequence $(\mathcal{F}_m^{\tilde{s}, \tilde{t}})_{m \in \omega}$ of finite nonempty subsets of A satisfying $\bigcup_{m \in \omega} \{b_m^{\tilde{s}, \tilde{t}}(x) : x \in \mathcal{F}_m^{\tilde{s}, \tilde{t}}\} \in \text{Dom}(Q)$. Now, for each $m \in \omega$, write $\mathcal{F}_m^{\tilde{s}, \tilde{t}} = \{a_{\tilde{s} \smallfrown (m), \tilde{t} \smallfrown (n)} : n \in \omega\}$ with each element of $\mathcal{F}_m^{\tilde{s}, \tilde{t}}$ appearing infinitely many times in the listing.

This completes the definition of $\langle a_{s,t} : (s,t) \in \mathcal{P} \setminus \{(\emptyset, \emptyset)\} \rangle$. We now claim that the family

$$\bigcup_{(s,t) \in \mathcal{P}} \bigcup_{m,n \in \omega} \left\{ (b_i^{l_{s,t}^{m,t}}, b_i^{l_{s,t}^{m,t}}) : i \in F_m^{s,t}(a_{s \smallfrown (m), t \smallfrown (n)}) \right\}$$

is an element of $\text{Dom}(Q \otimes Q')$.

Let then $(c, c') \in C \times C'$ be arbitrary. Recursively for each $h \in \omega$, choose $m_h, n_h \in \omega$ such that $cTb_{m_h}^{(m_k)_{k < h}, (n_k)_{k < h}}(a_{(m_k)_{k \leq h}, (n_k)_{k \leq h}})$. Since \preceq is upwards Q -compatible, this implies that $cTb_i^{(m_k)_{k < h}, (n_k)_{k < h}}$ for all $i \in F_{m_h}^{(m_k)_{k < h}, (n_k)_{k < h}}(a_{(m_k)_{k \leq h}, (n_k)_{k \leq h}})$; hence, it suffices to verify that, for some $h \in \omega$, there is $i \in F_{m_h}^{(m_k)_{k < h}, (n_k)_{k < h}}(a_{(m_k)_{k \leq h}, (n_k)_{k \leq h}})$ with $c'T'b_i^{(m_k)_{k < h}, (n_k)_{k < h}}$. Indeed, if this were not the case,

ONE	TWO
$Z_{l_{n_0}^{\emptyset, \emptyset}}^{a_{(m_0), (n_0)}}$	$\left\{ b_i^{l_{m_0}^{\emptyset, \emptyset}} : i \in F_{m_0}^{\emptyset, \emptyset}(a_{(m_0), (n_0)}) \right\}$
$Z_{l_{n_1}^{(m_0), (n_0)}}^{a_{(m_0), (m_1), (n_0), (n_1)}}$	$\left\{ b_i^{l_{m_1}^{(m_0), (n_0)}} : i \in F_{m_1}^{(m_0), (n_0)}(a_{(m_0), (m_1), (n_0), (n_1)}) \right\}$
\vdots	\vdots
$Z_{l_{n_h}^{(m_k)_{k \leq h}, (n_k)_{k \leq h}}}^{a_{(m_k)_{k \leq h}, (n_k)_{k \leq h}}}$	$\left\{ b_i^{l_{m_h}^{(m_k)_{k < h}, (n_k)_{k < h}}} : i \in F_{m_h}^{(m_k)_{k < h}, (n_k)_{k < h}}(a_{(m_k)_{k \leq h}, (n_k)_{k \leq h}}) \right\}$
\vdots	\vdots

would be a play of $G_{\text{fin}}(\text{Dom}(P'), \text{Dom}(Q'))$ in which TWO follows the winning strategy σ and loses.

(c) Analogous to the proof of Proposition 5.1. \square

The following consequence of Proposition 5.3 was originally proven by Telgársky under the extra assumption that the space is regular; it can be obtained by putting together Corollary 14.14 and Theorem 14.12 of [35] and Corollary 3 of [38]. Here we follow the standard terminology and refer to the game $G_{\text{fin}}(\mathcal{O}, \mathcal{O})$ as *the Menger game*; a topological space is *Menger* [24] if it satisfies $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$.

Corollary 5.4 (Telgársky [35, 38], for regular spaces). *If TWO has a winning strategy in the Menger game on a topological space X , then X is both productively Lindelöf and productively Menger.*

Proof. Let Y be a Lindelöf (respectively, Menger) space. Apply Proposition 5.3(a) (respectively, (b)) with $P' = Q' = (X, \tau, \subseteq)$, $P = Q = (Y, \rho, \subseteq)$ and $U \preceq V \leftrightarrow U \subseteq V$. \square

As in the observation made after Corollary 4.6, it is worth comparing Corollary 5.4 with Theorem 11(2) of [9].

It is also worth remarking that, assuming the Continuum Hypothesis, one can construct a Sierpiński set $S \subseteq \mathbb{R}$ such that $\mathbb{R} \setminus \mathbb{Q}$ is a continuous image of $S \times S$ [25, Theorem 2.12], which implies that $S \times S$ is not Menger. By Proposition 14 of [9], TWO has a winning strategy in the “length- $(\omega + \omega)$ ” variation of the Menger game played on every Sierpiński set. This shows that the existence of a winning strategy for TWO in this longer version of the Menger game, although stronger than the Menger property, is not strong enough to imply its productivity.

Corollary 5.5. *Let X be a topological space on which TWO has a winning strategy in the game $G_{\text{fin}}(\Omega, \Omega)$. Then:*

- (a) *if Y is a topological space such that Ω_Y -Lindelöf holds, then $\Omega_{X \times Y}$ -Lindelöf holds;*
- (a') *if Y is a topological space such that every finite power of Y is Lindelöf, then every finite power of $X \times Y$ is Lindelöf;*
- (b) *the product $X \times Y$ satisfies $S_{\text{fin}}(\Omega, \Omega)$ for every topological space Y satisfying $S_{\text{fin}}(\Omega, \Omega)$;*
- (b') *if Y is a topological space such that every finite power of Y is Menger, then every finite power of $X \times Y$ is Menger;*
- (c) *if TWO has a winning strategy in the game $G_{\text{fin}}(\Omega, \Omega)$ on a topological space Y , then he has a winning strategy in $G_{\text{fin}}(\Omega, \Omega)$ on the product $X \times Y$.*

Proof. First, we recall that a topological space Y is Lindelöf in every finite power if and only if Ω_Y -Lindelöf holds [20], and that it is Menger in every finite power if and only if $S_{\text{fin}}(\Omega_Y, \Omega_Y)$ holds [25, Theorem 3.9]. Now apply Proposition 5.3 with $P' = Q' = ([X]^{<\aleph_0}, \tau, \subseteq)$, $P = Q = ([Y]^{<\aleph_0}, \rho, \subseteq)$ and $U \preceq V \leftrightarrow U \subseteq V$, having in mind the observation made in the proof of Corollary 4.9. \square

For a topological space X , the space of all real-valued continuous functions on X regarded as a subspace of the (Tychonoff) power \mathbb{R}^X is denoted by $C_p(X)$. Recall that a topological space has *countable fan tightness* [3] at a point x if $S_{\text{fin}}(\Omega_x, \Omega_x)$ holds.

Corollary 5.6. *Let X and Y be completely regular spaces. Assume that TWO has a winning strategy in the game $G_{\text{fin}}(\Omega_0, \Omega_0)$ played on $C_p(X)$.*

- (a) *If $C_p(Y)$ is countably tight, then $C_p(X) \times C_p(Y)$ is also countably tight.*
- (b) *If $C_p(Y)$ has countable fan tightness, then the product $C_p(X) \times C_p(Y)$ also has countable fan tightness.*
- (c) *If TWO has a winning strategy in $G_{\text{fin}}(\Omega_0, \Omega_0)$ played on $C_p(Y)$, then TWO has a winning strategy in $G_{\text{fin}}(\Omega_{(0,0)}, \Omega_{(0,0)})$ played on the product $C_p(X) \times C_p(Y)$.*

Proof. This follows from Corollary 5.5, in view of the following results:

- for a completely regular space X , TWO has a winning strategy in $G_{\text{fin}}(\Omega, \Omega)$ on X if and only if TWO has a winning strategy in $G_{\text{fin}}(\Omega_0, \Omega_0)$ on $C_p(X)$ [34, Theorem 26];
- a completely regular space X is Menger in every finite power if and only if $C_p(X)$ has countable fan tightness [3, Theorem 4];
- a completely regular space X is Lindelöf in every finite power if and only if $C_p(X)$ is countably tight — see [2, Theorem 2] and [29, Theorem 1].

□

The hypothesis about the partial order \preceq in Proposition 5.3 is essential and there is no possibility of obtaining a result as general as Proposition 4.5 for properties involving finite selections. This can be seen, for example, by considering the same properties appearing in Corollary 5.6: It follows from a result of Uspenskiĭ [41, Theorem 1] that a space X is Lindelöf in the G_δ -topology if and only if $C_p(X)$ is productively countably tight; note that, by Theorem 26 of [34], TWO has a winning strategy in $G_{\text{fin}}(\Omega_0, \Omega_0)$ on $C_p(\mathbb{R})$ since TWO has a winning strategy in $G_{\text{fin}}(\Omega, \Omega)$ on \mathbb{R} [10, Lemma 2], and yet $C_p(\mathbb{R})$ is not productively countably tight in view of Uspenskiĭ's result.

6. γ -DOMINATING SEQUENCES

In this section we study games defined in terms of γ -dominating sequences, which allow us to express convergence-like properties.

We start off with a simple fact:

Lemma 6.1. *Let $P = (A, B, R)$ be a relation and $(z_n)_{n \in \omega} \in {}^\omega B$. The following assertions are equivalent:*

- (a) *$\{n \in \omega : \neg(aRz_n)\}$ is finite for all $a \in A$;*
- (b) *$\{z_n : n \in X\} \in \text{Dom}(P)$ for all $X \in [\omega]^{\aleph_0}$.*

Definition 6.2. *Let $P = (A, B, R)$ be a relation. A sequence $(z_n)_{n \in \omega} \in {}^\omega B$ is γ -dominating in P if the conditions in Lemma 6.1 hold. The family of all γ -dominating sequences in P will be denoted by $\text{Dom}_\gamma(P)$.*

We shall now consider variations of the games G and G_1 involving γ -dominating sequences.

Definition 6.3. *Let P and Q be relations.*

- (a) *The game $G_\gamma(P, Q)$ is played according to the same rules as $G(P, Q)$, but now ONE wins if $(b_n)_{n \in \omega} \in \text{Dom}_\gamma(Q)$, and loses otherwise.*

- (b) In a slight abuse of notation (cf. Definition 3.8), we shall designate by $G_1(\text{Dom}(P), \text{Dom}_\gamma(Q))$ the game played according to the same rules as $G_1(\text{Dom}(P), \text{Dom}(Q))$, but in which the winner is TWO if $(b_n)_{n \in \omega} \in \text{Dom}_\gamma(Q)$ and ONE otherwise.

The games defined above satisfy the following duality theorem, which parallels Theorem 3.10.

Theorem 6.4 (Galvin [18]). *The games $G_\gamma(P, Q)$ and $G_1(\text{Dom}(P), \text{Dom}_\gamma(Q))$ are dual for all relations P and Q .*

The following consequence of Theorem 6.4 was observed in the proof of Proposition 1 of [19]. First, let us recall the game $G_{O,P}(X, x)$, introduced by Gruenhage in [21]. Let a topological space X and $x \in X$ be fixed. In each inning $n \in \omega$, ONE picks $V_n \in \tau_x$, and then TWO chooses $x_n \in V_n$. ONE wins if the sequence $(x_n)_{n \in \omega}$ converges to x , and loses otherwise.

Corollary 6.5 (Gerlits [19]). *Let X be a topological space, $x \in X$ and $\Gamma_x = \{(x_n)_{n \in \omega} \in {}^\omega X : x_n \xrightarrow{n \rightarrow \infty} x\}$. Then the game $G_{O,P}(X, x)$ and the game $G_1(\Omega_x, \Gamma_x)$ on X are dual.*

Proof. Apply Theorem 6.4 with $P = Q = (\tau_x, X, \ni)$. \square

Our main goal in this section is to find conditions under which the existence of a winning strategy for ONE in the game $G(P, Q)$ yields the existence of a winning strategy for ONE also in the game $G_\gamma(P, Q)$. In order to formulate such conditions, we will need the following auxiliary notion.

Definition 6.6 (cf. Definition 5.2). *Let $P = (A, B, R)$ be a relation. We say that a partial order \leq on A is downwards P -compatible if, for all $a_1, a_2 \in A$ and $b \in B$,*

$$(a_1 \leq a_2 \ \& \ a_2 R b) \rightarrow a_1 R b.$$

The argument for the next result is essentially taken from Theorem 3.9 of [21] (which we state as Corollary 6.8); see also Theorem 1 of [20] (also stated here as Corollary 6.10).

Theorem 6.7 (Gruenhage [21]). *Let $P = (A, B, R)$ and $Q = (C, D, T)$ be relations. Suppose that there is a downwards P -compatible partial order \leq on A such that, for each finite subset F of A , there is $\tilde{a}(F) \in A$ satisfying $a \leq \tilde{a}(F)$ for all $a \in F$. Then the following conditions are equivalent:*

- (a) ONE has a winning strategy in $G(P, Q)$;
- (b) ONE has a winning strategy in $G_\gamma(P, Q)$.

Proof. The implication (b) \rightarrow (a) is immediate. We will prove that (a) implies (b).

Let $\sigma : {}^{<\omega}B \rightarrow A$ be a winning strategy for ONE in $G(P, Q)$. For each $n \in \omega$, let S_n be the (finite) set of all strictly increasing sequences with range included in n . Now define $\varphi : {}^{<\omega}B \rightarrow A$ by $\varphi((b_j)_{j < n}) = \tilde{a}(\{\sigma((b_{s(i)})_{i \in \text{dom}(s)}) : s \in S_n\})$. We claim that φ is a winning strategy for ONE in $G_\gamma(P, Q)$.

Indeed, let $(a_0, b_0, a_1, b_1, \dots, a_n, b_n, \dots)$ be a play of $G_\gamma(P, Q)$ in which ONE follows the strategy φ , and let $X \in [\omega]^{\aleph_0}$ be arbitrary. Write $X = \{x_k : k \in \omega\}$ with $x_k < x_{k+1}$ for all $k \in \omega$. As \leq is downwards P -compatible, it follows that

$$(\sigma(\emptyset), b_{x_0}, \sigma((b_{x_0})), b_{x_1}, \sigma((b_{x_0}, b_{x_1})), b_{x_2}, \dots, \sigma((b_{x_i})_{i < k}), b_{x_k}, \dots)$$

is a play of $\mathbf{G}(P, Q)$ in which ONE makes use of the winning strategy σ , since for each $k \in \omega$ we have

$$\sigma((b_{x_i})_{i < k}) \leq \tilde{a}(\{\sigma((b_{s(i)})_{i \in \text{dom}(s)}) : s \in S_{x_k}\}) = \varphi((b_j)_{j < x_k}) = a_{x_k}$$

and $a_{x_k} R b_{x_k}$. Thus $\{b_{x_k} : k \in \omega\} \in \text{Dom}(Q)$; since X was chosen arbitrarily, it follows that $(b_n)_{n \in \omega} \in \text{Dom}_\gamma(Q)$. \square

Corollary 6.8 (Gruenhage [21]). *Let X be a topological space and $x \in X$. The following statements are equivalent:*

- (a) *ONE has a winning strategy in the game $\mathbf{G}_{O,P}^c(X, x)$ (see Example 3.6);*
- (b) *ONE has a winning strategy in the game $\mathbf{G}_{O,P}(X, x)$ (see paragraph preceding Corollary 6.5).*

Proof. Apply Theorem 6.7 with $P = Q = (\tau_x, X, \ni)$ and $V \leq W \leftrightarrow V \supseteq W$. \square

Before stating the next result, we evoke Corollaries 4.3 and 4.4 of [35], which can be combined in a single statement as:

Proposition 6.9 (Telgársky [35]). *The point-open game is equivalent to the finite-open game, which is played according to the following rules: In each inning $n \in \omega$, ONE picks a finite subset F_n of X , and then TWO chooses $U_n \in \tau$ with $F_n \subseteq U_n$; ONE wins if $\{U_n : n \in \omega\} \in \mathcal{O}_X$, and TWO wins otherwise.*

With Proposition 6.9 in mind, we recall the *strict point-open game*, introduced in [20]. The game is played according to the same rules as the finite-open game,⁵ but now ONE wins if $X = \bigcup_{n \in \omega} \bigcap_{m \in \omega \setminus n} U_m$.

Corollary 6.10 (Gerlits–Nagy [20]). *Let X be a topological space. The following statements are equivalent:*

- (a) *ONE has a winning strategy in the point-open game on X ;*
- (b) *ONE has a winning strategy in the strict point-open game on X .*

Proof. Apply Theorem 6.7 with $P = ([X]^{<\aleph_0}, \tau, \subseteq)$, $Q = (X, \tau, \in)$ and $F \leq G \leftrightarrow F \subseteq G$. \square

For the next two corollaries, we recall the game $\mathbf{G}^*(X)$ introduced by Gruenhage in [22] for every noncompact space X . In each inning $n \in \omega$ of $\mathbf{G}^*(X)$, ONE picks a compact set $C_n \subseteq X$, and then TWO picks a nonempty compact set $L_n \subseteq X$ with $C_n \cap L_n = \emptyset$. ONE wins if the family $\{L_n : n \in \omega\}$ is locally finite, and loses otherwise. We shall also consider a variation $\mathbf{G}^{**}(X)$ of this game, which is played according to the same rules but now ONE wins if and only if $\{L_n : n \in \omega\} \in \mathfrak{M}_X$.

Corollary 6.11. *Let X be a noncompact locally compact space. The following assertions are equivalent:*

- (a) *ONE has a winning strategy in $\mathbf{G}^*(X)$;*
- (b) *ONE has a winning strategy in $\mathbf{G}^{**}(X)$.*

Proof. For (b) \rightarrow (a), let σ be a winning strategy for ONE in $\mathbf{G}^{**}(X)$. We can define a strategy φ for ONE in $\mathbf{G}^*(X)$ by setting $\varphi((L_i)_{i < n}) = \sigma((L_i)_{i < n}) \cup \bigcup_{i < n} L_i$ for all $(L_i)_{i < n} \in {}^{<\omega}(K(X) \setminus \{\emptyset\})$. Note that φ is a winning strategy since the set of

⁵In view of Proposition 6.9, the authors make no distinction between the point-open game and the finite-open game in [20].

TWO's moves in a play in which ONE follows φ is an infinite locally finite family of nonempty compact sets, and hence is a moving-off family by Lemma 4 of [8].

For $(a) \rightarrow (b)$, apply Theorem 6.7 with $P = Q = (K(X), K(X) \setminus \{\emptyset\}, R)$, where $CRL \leftrightarrow C \cap L = \emptyset$. The result follows from the observation that, if X is locally compact and $(L_n)_{n \in \omega} \in \text{Dom}_\gamma(Q)$, then $\{L_n : n \in \omega\}$ is locally finite. \square

The next corollary presents some variations on the game-theoretic characterization of paracompactness for locally compact T_2 spaces obtained by Gruenhage in [22] — which states that a locally compact T_2 space is paracompact if and only if ONE has a winning strategy in the game $G^*(X)$.

Corollary 6.12. *Let X be a noncompact locally compact T_2 space and $\mathfrak{L} = \{\mathcal{L} \subseteq K(X) \setminus \{\emptyset\} : \mathcal{L} \text{ is locally finite}\}$. The following conditions are equivalent:*

- (a) X is paracompact;
- (b) ONE has a winning strategy in $G^*(X)$;
- (c) ONE has a winning strategy in $G^{**}(X)$;
- (d) TWO has a winning strategy in $G_1(\mathfrak{M}, \mathfrak{L})$;
- (e) TWO has a winning strategy in $G_1(\mathfrak{M}, \mathfrak{M})$.

Proof. We have just proven $(b) \leftrightarrow (c)$ in Corollary 6.11. The equivalences $(a) \leftrightarrow (b)$ and $(b) \leftrightarrow (d)$ are Theorem 5 of [22] and Theorem 2 of [8] respectively. Finally, $(c) \leftrightarrow (e)$ follows from Theorem 3.10 with $P = Q = (K(X), K(X) \setminus \{\emptyset\}, R)$, where $CRL \leftrightarrow C \cap L = \emptyset$. \square

7. \aleph_0 -MODIFICATIONS

In this section, we study another variation of the game $G(P, Q)$ (resp. $G_1(\text{Dom}(P), \text{Dom}(Q))$) for which the existence of a winning strategy for player ONE (resp. TWO) although apparently stronger, turns out to be equivalent to the same condition for the original game.

This variation will be defined in terms of the following concept.

Definition 7.1. *The \aleph_0 -modification of a relation $P = (A, B, R)$ is the relation $P_{\aleph_0} = (A, [B]^{\leq \aleph_0} \setminus \{\emptyset\}, \tilde{R})$, where $a\tilde{R}E \leftrightarrow \forall b \in E (aRb)$.*

The equivalence previously mentioned can then be stated as follows.

Proposition 7.2. *Let P and Q be relations. The following conditions are equivalent:*

- (a) ONE has a winning strategy in the game $G(P, Q)$;
- (b) ONE has a winning strategy in the game $G(P_{\aleph_0}, Q_{\aleph_0})$.

Proof. Write $P = (A, B, R)$ and $Q = (C, D, T)$.

The implication $(b) \rightarrow (a)$ is immediate, since $G(P, Q)$ is equivalent to the game $G(P_{\aleph_0}, Q_{\aleph_0})$ played with the additional restriction that TWO must choose one-element subsets of B .

For $(a) \rightarrow (b)$, let $\sigma : {}^{<\omega}B \rightarrow A$ be a winning strategy for ONE in $G(P, Q)$. Fix an injective function $s \mapsto m_s$ from ${}^{<\omega}\omega$ to ω satisfying $s \subseteq t \rightarrow m_s \leq m_t$ for every $s, t \in {}^{<\omega}\omega$ — for example, define $m_s = \prod_{i \in \text{dom}(s)} p_i^{s(i)+1}$, where p_i is the i -th prime number. Now write each $E \in [B]^{\leq \aleph_0} \setminus \{\emptyset\}$ as $E = \{b_k^E : k \in \omega\}$, and let $\tilde{a} \in A$ be

fixed. Define $\varphi : {}^{<\omega}([B]^{\leq \aleph_0} \setminus \{\emptyset\}) \rightarrow A$ by

$$\varphi((E_j)_{j<n}) = \begin{cases} \sigma \left(\left(b_{s(i)}^{E_{m_s \upharpoonright i}} \right)_{i \in \text{dom}(s)} \right), & \text{if } n = m_s; \\ \tilde{a}, & \text{otherwise.} \end{cases}$$

We claim that φ is a winning strategy for ONE in $\mathbf{G}(P_{\aleph_0}, Q_{\aleph_0})$.

Indeed, let $(a_0, E_0, a_1, E_1, \dots)$ be a play of $\mathbf{G}(P_{\aleph_0}, Q_{\aleph_0})$ in which ONE follows the strategy φ , and suppose that there is $c \in C$ such that $c \tilde{T} E_n$ does not hold for any $n \in \omega$. Define $f : \omega \rightarrow \omega$ by recursively choosing $f(i) \in \omega$ such that $c T b_{f(i)}^{E_{m_f \upharpoonright i}}$ does not hold. Then we obtain a contradiction from the fact that

ONE	TWO
$\sigma(\emptyset)$	$b_{f(0)}^{E_{m_\emptyset}}$
$\sigma \left(\left(b_{f(0)}^{E_{m_\emptyset}} \right) \right)$	$b_{f(1)}^{E_{m_{(f(0))}}}$
$\sigma \left(\left(b_{f(0)}^{E_{m_\emptyset}}, b_{f(1)}^{E_{m_{(f(0))}}} \right) \right)$	$b_{f(2)}^{E_{m_{(f(0), f(1))}}}$
\vdots	\vdots
$\sigma \left(\left(b_{f(j)}^{E_{m_f \upharpoonright j}} \right)_{j < i} \right)$	$b_{f(i)}^{E_{m_f \upharpoonright i}}$
\vdots	\vdots

is a play of $\mathbf{G}(P, Q)$ in which ONE makes use of the winning strategy σ and loses — since none of TWO's moves dominate c in T . \square

In view of Theorem 3.10, we can rewrite Proposition 7.2 as:

Corollary 7.3. *Let P and Q be relations. The following conditions are equivalent:*

- (a) *TWO has a winning strategy in the game $\mathbf{G}_1(\text{Dom}(P), \text{Dom}(Q))$;*
- (b) *TWO has a winning strategy in the game $\mathbf{G}_1(\text{Dom}(P_{\aleph_0}), \text{Dom}(Q_{\aleph_0}))$.*

The following result is Theorem 5.1 of [36]. Given a nonempty family \mathbf{K} of subsets of a topological space X , we call **\mathbf{K} -open game on X** the game in which, in each inning $n \in \omega$, ONE chooses $K_n \in \mathbf{K}$ and then TWO picks an open set $U_n \subseteq X$ with $K_n \subseteq U_n$; the winner is ONE if $X = \bigcup_{n \in \omega} U_n$, and TWO otherwise. The **\mathbf{K} - G_δ game on X** is played according to the same rules, replacing “open” with “ G_δ ”.

Corollary 7.4 (Telgársky [36]). *Let X be a topological space and \mathbf{K} be a nonempty family of subsets of X . The following conditions are equivalent:*

- (a) ONE has a winning strategy in the \mathbf{K} -open game on X ;
- (b) ONE has a winning strategy in the \mathbf{K} - G_δ game on X .

Proof. Apply Proposition 7.2 with $P = (\mathbf{K}, \tau, \subseteq)$ and $Q = (X, \tau, \in)$. The result follows from the observation that the games \mathbf{K} - G_δ and $\mathbf{G}(P_{\aleph_0}, Q_{\aleph_0})$ are equivalent. \square

As another consequence of Proposition 7.2, we have:

Corollary 7.5. *Let X be a topological space and $x \in X$. The following conditions are equivalent:*

- (a) TWO has a winning strategy in the game $\mathbf{G}_1(\Omega_x, \Omega_x)$;
- (b) TWO has a winning strategy in the game $\mathbf{G}_1(\pi\mathcal{N}_x, \pi\mathcal{N}_x)$;
- (c) TWO has a winning strategy in the game $\mathbf{G}_1(\pi\mathcal{N}_x^{\aleph_0}, \pi\mathcal{N}_x^{\aleph_0})$.

Proof. It is clear that (c) \rightarrow (b) \rightarrow (a). Now the equivalence between (a) and (c) follows from Corollary 7.3 with $P = Q = (\tau_x, X, \ni)$. \square

As an immediate consequence of Corollary 7.5 (see also Corollary 8.6), we have the following result, which answers Question 4.9 of [4] in the affirmative.

Corollary 7.6. *Let X be a topological space and $x \in X$. If TWO has a winning strategy in the game $\mathbf{G}_1(\Omega_x, \Omega_x)$ on X , then $\pi\mathcal{N}_x^{\aleph_0}$ -Lindelöf holds.*

8. \aleph_0 -PRESERVING RELATIONS

Inspired by some features of relations of the form P_{\aleph_0} introduced in the previous section (see Lemma 8.3), we will now aim at finding general conditions on relations under which a Lindelöf-like property turns out to be strong enough to yield the nonexistence of a winning strategy for ONE in the associated selective game.

Definition 8.1. *Let $P = (A, B, R)$ be a relation and \preceq be a partial order on B . We say that \preceq is countably downwards P -compatible if, for every $a \in A$ and every $E \in [B]^{\leq \aleph_0} \setminus \{\emptyset\}$,*

$$(\forall b \in E (aRb)) \rightarrow \exists \tilde{b} \in B (aR\tilde{b} \ \& \ \forall b \in E (\tilde{b} \preceq b)).$$

Definition 8.2. *A relation $P = (A, B, R)$ is \aleph_0 -preserving if there is a partial order \preceq on B that is both upwards P -compatible (see Definition 5.2) and countably downwards P -compatible.*

Our main examples of \aleph_0 -preserving relations will be of the form P_{\aleph_0} :

Lemma 8.3. *Let P be a relation. Then P_{\aleph_0} (see Definition 7.1) is an \aleph_0 -preserving relation.*

Proof. Just note that, if $P = (A, B, R)$, then the partial order \preceq on $[B]^{\leq \aleph_0}$ defined by $E_1 \preceq E_2 \leftrightarrow E_1 \supseteq E_2$ witnesses the fact that P_{\aleph_0} is \aleph_0 -preserving. \square

The following proposition is the main result of this section.

Proposition 8.4. *Let P be an \aleph_0 -preserving relation. The following conditions are equivalent:*

- (a) $\text{Dom}(P)$ -Lindelöf;
- (b) $\mathbf{S}_1(\text{Dom}(P), \text{Dom}(P))$;
- (c) ONE does not have a winning strategy in the game $\mathbf{G}_1(\text{Dom}(P), \text{Dom}(P))$.

Proof. Clearly, $(c) \rightarrow (b) \rightarrow (a)$. We will prove the implication $(a) \rightarrow (c)$.

Let a strategy for ONE in $G_1(\text{Dom}(P), \text{Dom}(P))$ be fixed. By (a), we may assume that each of ONE's moves in this strategy is a countable set; this allows us to regard such strategy as an indexed family $(b_t)_{t \in {}^{<\omega}\omega \setminus \{\emptyset\}}$ — meaning that, if $s \in {}^n\omega$ is such that TWO's choices in the first n innings were $(b_{s \upharpoonright i})_{i < n}$, then ONE's move in the n -th inning is $\{b_{s \frown (k)} : k \in \omega\} \in \text{Dom}(P)$.

Now write $P = (A, B, R)$, and let \preceq be a partial order on B witnessing the fact that P is \aleph_0 -preserving. For each $a \in A$ and each function $F : {}^{<\omega}\omega \rightarrow \omega$, let then $\tilde{b}_a^F \in B$ be such that $aR\tilde{b}_a^F$ and $\forall s \in {}^{<\omega}\omega$ ($\tilde{b}_a^F \preceq b_{s \frown (F(s))}$). Note that $\{\tilde{b}_a^F : a \in A, F \in ({}^{<\omega}\omega)^\omega\} \in \text{Dom}(P)$; by (a), it follows that there exist $\{a_n : n \in \omega\} \subseteq A$ and $\{F_n : n \in \omega\} \subseteq ({}^{<\omega}\omega)^\omega$ with $\{\tilde{b}_{a_n}^{F_n} : n \in \omega\} \in \text{Dom}(P)$. Now define $f : \omega \rightarrow \omega$ recursively by $f(n) = F_n(f \upharpoonright n)$ for each $n \in \omega$. We claim that $\{b_{f \upharpoonright (n+1)} : n \in \omega\} \in \text{Dom}(P)$ — which shows that the strategy at hand for ONE can be defeated.

In order to see this, let $a \in A$ be arbitrary. Since $\{\tilde{b}_{a_n}^{F_n} : n \in \omega\} \in \text{Dom}(P)$, there is $m \in \omega$ such that $aR\tilde{b}_{a_m}^{F_m}$; thus, as $\tilde{b}_{a_m}^{F_m} \preceq b_{s \frown (F(s))}$ holds for $s = f \upharpoonright m$ in particular, it follows from the equality $f(m) = F_m(f \upharpoonright m)$ and the hypothesis that \preceq is upwards P -compatible that $aRb_{f \upharpoonright (m+1)}$, as required. \square

A topological space is *strongly Alster* [5] if \mathcal{G}_K -Lindelöf holds, where $\mathcal{G}_K = \{\mathcal{W} : (\forall W \in \mathcal{W} (W \text{ is a } G_\delta \text{ subset of } X)) \ \& \ (\forall C \in K(X) \exists W \in \mathcal{W} (C \subseteq W))\}$.

Corollary 8.5. *The following statements are equivalent for a topological space X :*

- (a) X is strongly Alster;
- (b) $S_1(\mathcal{G}_K, \mathcal{G}_K)$;
- (c) ONE does not have a winning strategy in the game $G_1(\mathcal{G}_K, \mathcal{G}_K)$.

Proof. Apply Proposition 8.4 with $P = (K(X), G_\delta(X), \subseteq)$, where $K(X) = \{C \subseteq X : C \text{ is compact}\}$ and $G_\delta(X) = \{W \subseteq X : W \text{ is a countable intersection of open sets}\}$. (Note that $W_1 \preceq W_2 \leftrightarrow W_1 \subseteq W_2$ witnesses that P is \aleph_0 -preserving.) \square

The next corollary deals with a game that was also explored in Corollary 7.5.

Corollary 8.6. *Let X be a topological space and $x \in X$. The following conditions are equivalent:*

- (a) $\pi\mathcal{N}_x^{\aleph_0}$ -Lindelöf holds;
- (b) $S_1(\pi\mathcal{N}_x^{\aleph_0}, \pi\mathcal{N}_x^{\aleph_0})$;
- (c) ONE does not have a winning strategy in the game $G_1(\pi\mathcal{N}_x^{\aleph_0}, \pi\mathcal{N}_x^{\aleph_0})$.

Proof. Note that $P = (\tau_x, [X]^{\leq \aleph_0}, \supseteq)$ is \aleph_0 -preserving by Lemma 8.3. Now apply Proposition 8.4. \square

We note that the equivalence between (a) and (b) in Corollary 8.6 also follows from Proposition 2.5(2) of [11].

9. REMARKS

A. As not all topological properties can be expressed in terms of relations, it should be made clear that there are selective topological games that have been studied in the literature for which the analogue of the previous results does not hold. We illustrate this with the following selective property: A topological space X is *selectively screenable* [1] if, for every sequence $(\mathcal{U}_n)_{n \in \omega}$ of open covers of X , there is

a sequence $(\mathcal{V}_n)_{n \in \omega}$ of families of open subsets of X such that $X = \bigcup_{n \in \omega} \mathcal{V}_n$ and each \mathcal{V}_n is a pairwise disjoint partial refinement of \mathcal{U}_n . It follows from Example 1 of [28] and Theorem 2.2 of [7] that TWO having a winning strategy in the game naturally associated with selective screenability does not imply that the space is productively selectively screenable; therefore, a result similar to Propositions 4.5 and 5.3 could not be obtained for this concept.

B. Having in mind the properties E_1 – E_4 from the Introduction, there seems to be a gap in Propositions 4.5 and 5.3, which motivates the two main open questions of this paper:

Problem 9.1. *Let P, P', Q and Q' be relations such that TWO has a winning strategy in the game $G_1(\text{Dom}(P'), \text{Dom}(Q'))$ and ONE does not have a winning strategy in the game $G_1(\text{Dom}(P), \text{Dom}(Q))$. Does it follow that ONE does not have a winning strategy in the game $G_1(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$?*

Problem 9.2. *Let $P = (A, B, R)$, $P' = (A', B', R')$, $Q = (C, D, T)$ and $Q' = (C', D', T')$ be relations with $B = D$ such that:*

- *there is a partial order \preceq on B that is both downwards P -compatible and upwards Q -compatible;*
- *TWO has a winning strategy in the game $G_{\text{fin}}(\text{Dom}(P'), \text{Dom}(Q'))$; and*
- *ONE does not have a winning strategy in the game $G_{\text{fin}}(\text{Dom}(P), \text{Dom}(Q))$.*

Does it follow that ONE does not have a winning strategy in the game $G_{\text{fin}}(\text{Dom}(P \otimes P'), \text{Dom}(Q \otimes Q'))$?

It should be pointed out that, in many instances of the topological properties in which we are interested in this paper, it is the case that ONE does not have a winning strategy in the game $G_1(\text{Dom}(P), \text{Dom}(Q))$ if and only if $S_1(\text{Dom}(P), \text{Dom}(Q))$ holds (and similarly for G_{fin} and S_{fin}); see e.g. [24, Theorem 10] (for the Menger game), [27, Lemma 2] (for the Rothberger game) and [32, Theorems 2 and 14] (for the games $G_{\text{fin}}(\mathcal{D}, \mathcal{D})$ and $G_1(\mathcal{D}, \mathcal{D})$). As a consequence, none of these instances could provide us with a negative answer to Problems 9.1 and 9.2.

It is also known that there are other instances in which this equivalence does not hold — such as $G_1(\mathfrak{D}, \mathfrak{D})$ [33, Example 3] —, which could be a first attempt to answer Problem 9.1 in the negative. More explicitly:

Problem 9.3. *Let X and Y be topological spaces such that TWO has a winning strategy in the game $G_1(\mathfrak{D}_X, \mathfrak{D}_X)$ and ONE does not have a winning strategy in the game $G_1(\mathfrak{D}_Y, \mathfrak{D}_Y)$. Does it follow that ONE does not have a winning strategy in the game $G_1(\mathfrak{D}_{X \times Y}, \mathfrak{D}_{X \times Y})$?*

C. Regarding the four properties E_i mentioned in the Introduction, and having in mind Proposition 4.5, one could ask whether, for some pair (i, j) with $2 \leq i \leq j \leq 4$, it is the case that every relation in the class E_i is productively E_j . This possibility can be ruled out by considering the following.

For a topological space X and $P = Q = (X, \tau, \in)$, we have:

- E_2 : ONE does not have a winning strategy in the game $G_1(\mathcal{O}_X, \mathcal{O}_X)$;
- E_3 : $S_1(\mathcal{O}_X, \mathcal{O}_X)$;
- E_4 : X is a Lindelöf space.

By Lemma 2 of [27], E_2 and E_3 are equivalent. In Theorem 8 of [40], two Rothberger spaces are constructed in such a way that their product is not Lindelöf. The

conjunction of these results shows that E_2 is not strong enough to imply productivity with respect to either E_2 , E_3 or E_4 .

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